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Hardy-Sobolev type inequalities on homogeneous groups and applications

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CONTENTS

1 INTRODUCTION

1.1 Overview

The classical multidimensional L^2-Hardy inequality (in the Euclidean space $\mathbb{R}^n$) asserts that for $n \geq 3$

$$\int_{\mathbb{R}^n} |\nabla u|^2 dx \geq \left|\frac{n-2}{2}\right|^2 \int_{\mathbb{R}^n} \frac{|u|^2}{|x|^2} dx, \ \forall u \in C_0^\infty(\mathbb{R}^n), \tag{1.1}$$

where the constant $\left|\frac{n-2}{2}\right|^2$ is sharp, ∇ is the usual Euclidean gradient, and $|x| = \sqrt{x_1^2 + \cdots + x_n^2}$.

It has many applications in different areas such as the spectral theory, that leads to the lower bounds for the quadratic form associated with the Laplace operator, for instance. It is also related to many other fields, for example, the notable one is the uncertainty principle. The uncertainty principle in physics is a fundamental concept going back to Heisenberg's work on quantum mechanics [1, 2] as well as to its mathematical justification by Hermann Weyl [3]. In the Euclidean setting for all real-valued functions $u \in C_0^\infty(\mathbb{R}^n)$ it can be defined as

$$\left(\int_{\mathbb{R}^n} |\nabla u|^2 dx\right)\left(\int_{\mathbb{R}^n} |x|^2 |u|^2 dx\right) \geq \frac{n^2}{4}\left(\int_{\mathbb{R}^n} |u|^2 dx\right)^2, \tag{1.2}$$

where $n^2/4$ is optimal. Inequality (1.2) is a direct consequence of inequality (1.1). There are well-known surveys on the mathematical aspects of uncertainty principles by Fefferman [4] and by Folland and Sitaram [5]. Note that Ciatti, Ricci and Sundari [6] showed the uncertainty principle can be also derived without the Hardy inequalities.

One of the interesting extensions of the Hardy inequality is the so-called Rellich inequality, which was introduced by Rellich [7] which is in the form

$$\int_{\mathbb{R}^n} |\Delta u|^2 dx \geq \frac{n^2(n-4)^2}{16} \int_{\mathbb{R}^n} \frac{|u|^2}{|x|^4} dx, \ n \geq 5, \tag{1.3}$$

where the constant is sharp. We can refer, for instance, to Davies and Hinz [8] (see also Brézis and Vázquez [9]) for further history and later extensions, including the derivation of sharp constants.

There is now a whole scope of related inequalities playing fundamental roles in different branches of mathematics, in particular, in the theory of linear and nonlinear partial differential equations. For instance, the analysis of more general weighted Hardy-Sobolev type inequalities has also a long history, initiated by Caffarelli, Kohn and Nirenberg [10] as well as by Brézis and Nirenberg in [11], and then Brézis and Lieb [12] with a mixture with Sobolev inequalities, Brézis and Vázquez in Section 4 [9], [13] with many subsequent works in this direction. On this subject, we refer to the work of Hoffmann-Ostenhof and Laptev [14] and to references therein.

The pioneers of the subelliptic ideas of the analysis on the Heisenberg group were Folland and Stein [15], and consistent development of this theory by Folland [16] had led to the foundations for analysis on the stratified groups (or homogeneous Carnot groups). Furthermore, Rothschild and Stein extended this result for general Hörmander's operators. Moreover, the fundamental book [17] in 1982 titled *Hardy spaces on homogeneous groups*, by Folland and Stein laid down foundations for the 'anisotropic' analysis on general homogeneous groups, i.e. Lie groups equipped with a compatible family of dilations. Such groups are necessarily nilpotent, and the class of homogeneous groups almost covers the whole class of nilpotent Lie groups including the classes of stratified, and more generally, graded groups.

The study of the subelliptic functional estimates has also begun more than couple decades ago due to their importance for many questions involving subelliptic partial differential equations, unique continuation, sub-Riemannian geometry, subelliptic spectral theory, etc. As expected, for the first time, the subelliptic Hardy inequality was obtained to the most important example of the Heisenberg group by Garofalo and Lanconelli [18].

In recent years, the subelliptic functional inequalities and related analysis on the homogeneous groups have been a topic of intensive research summarised in the very recent appearing book titled *Hardy inequalities on homogeneous groups* by Ruzhansky and Suragan [19]. This book covers the most recent developments of the subelliptic functional inequalities on the Heisenberg groups, the stratified Lie groups, the graded Lie groups, and the homogeneous groups. Honourably, I would like to emphasize that some of my works with Ruzhansky and Suragan are also included in this book, and listed as the contributors for this book.

This PhD thesis is devoted to studying the research developments at the intersection of two subjects such as Hardy inequalities and the noncommutative analysis in the setting of the stratified Lie groups (homogeneous Carnot groups). More broad details of this theme can be founded the recent 'International award-winning' book called 'Hardy inequalities on Homogeneous Groups' by Ruzhansky and Suragan. Topics treated in this PhD thesis as follow:

1 Geometric Hardy and Hardy-Sobolev inequalities on the stratified groups. In this direction, we study the geometric Hardy and Hardy-Sobolev inequalities on the half-spaces by the formula

$$\int_{\mathbb{H}^+} |\nabla_H u|^2 d\xi \geq \frac{1}{4}\int_{\mathbb{H}^+} \frac{\mathcal{W}(\xi)^2}{dist(\xi,\partial\mathbb{H}^+)^2}|u|^2 d\xi,$$

and

$$\left(\int_{\mathbb{H}^+} |\nabla_H u|^2 d\xi - \frac{1}{4}\int_{\mathbb{H}^+} \frac{\mathcal{W}(\xi)^2}{dist(\xi,\partial\mathbb{H}^+)^2}|u|^2 d\xi\right)^{1/2} \geq C\left(\int_{\mathbb{H}^+} |u|^2 d\xi\right)^{\frac{Q-2}{2Q}},$$

where $dist(\xi,\partial\mathbb{H}^+)$ is the Euclidean distance to the boundary and the angle function

$$\mathcal{W}(\xi) := \left(\sum_{i=1}^{n} \langle X_i(\xi), \nu\rangle^2 + \langle Y_i(\xi), \nu\rangle^2\right)^{\frac{1}{2}}.$$

As a result, we prove the conjecture in the paper [20]. Also, we present L^2 and L^p versions of the (subelliptic) geometric Hardy inequalities in half-spaces and convex domains on general stratified groups. As a consequence, we have derived the Hardy-Sobolev inequality in the half-space on the Heisenberg group [21]. Moreover, the geometric Hardy inequality on th starshaped sets is established in [22].

2 Horizontal inequalities on the stratified groups. In the second direction, we study the following horizontal version of Hardy type inequalities

$$\int_{\mathbb{G}} |\nabla_H u(x)|^2 dx \geq \left(\frac{N-2}{2}\right)^2 \int_{\mathbb{G}} \frac{|u(x)|^2}{|x'|^2} dx,$$

where $|\cdot|$ is the Euclidean norm and ∇_H is a horizontal gradient. As a result, the version of horizontal weighted Hardy-Rellich type inequalities was obtained on the stratified Lie groups as a consequence of this inequality Sobolev type spaces are defined on stratified Lie groups and the embedding theorems are proved for these functional spaces. Also, we have obtained the subelliptic Picone type identities, as a result, we proved the Hardy and Rellich type inequalities for the anisotropic p-sub-Laplacians. Moreover, analogues of Hardy type inequalities with multiple singularities and many-particle Hardy type inequalities are obtained on the stratified groups, [23, 24].

3 Hardy and Rellich type inequalities and the sub-Laplacian fundamental solutions. In the third direction, we investigate the following type of Hardy inequalities

$$\int_{\mathbb{G}} |\nabla_H u(x)|^2 \geq \left(\frac{Q-2}{2}\right)^2 \int_{\mathbb{G}} \frac{|u(x)|^2}{(d(x))^2} dx,$$

where Q is the homogeneous dimension of the stratified group $\mathbb{G}$ and $d(x)$ is the so-called $\mathcal{L}$-gauge, which is a particular homogeneous quasi-norm obtained from the fundamental solution of the sub-Laplacian. As a result, generalised weighted L^p-Hardy, L^p-Rellich, and L^p-Caffarelli-Kohn-Nirenberg type inequalities with boundary terms are obtained on the stratified groups. As consequences, most of the Hardy type inequalities and the Heisenberg-Pauli-Weyl type uncertainty principles on the stratified groups are recovered. Moreover, a weighted L^2-Rellich type inequality with the boundary term is obtained. We also present Hardy and Rellich inequalities for the sub-Laplacians in terms of their fundamental solutions on the quaternion Heisenberg group [25, 26].

4 Weighted Hardy and Rellich type inequalities for general vector fields. In this direction, we study the weighted Hardy and Rellich inequalities for general vector fields without a group structure as

$$\int_\Omega W(x)|\nabla_X u|^p dx \geq \int_\Omega H(x)|u|^p dx,\ \forall u \in C_0^1(\Omega),$$

and

$$\int_\Omega W(x)|\mathcal{L}_X u|^p dx \geq \int_\Omega H(x)|u|^p dx,\ \forall u \in C_0^\infty(\Omega).$$

Here, we establish the weighted anisotropic Hardy and Rellich type inequalities with boundary terms for general (real-valued) vector fields. As consequences, we derive new as well as many of the fundamental Hardy and Rellich type inequalities which are known in different settings [27].

Main results were published in the following journals:

1 Sabitbek B., Suragan D. Horizontal Weighted Hardy–Rellich Type Inequalities on Stratified Lie Groups // Complex Analysis and Operator Theory. – 2018, -V. 12, - P. 1469-1480. (Scopus, Web of Science, Q2)

2 Sabitbek B., Suragan D. On green functions for Dirichelet sub-Laplacians on a Quaternion Heisenberg group // Mathematical Modelling of Natural Phenomena. – 2018, –V. 13, - No. 4. (Scopus, Web of Science, Q3)

3 Ruzhansky M. Sabitbek B., Suragan D. Weighted Lp-Hardy and Lp-Rellich inequalities with boundary terms on stratified Lie groups // Revista Matematica Complutense. – 2019, – Vol. 32, - Issue 1, - P. 19–35. (Scopus, Web of Science, Q1)

4 Ruzhansky M. Sabitbek B., Suragan D. Weighted anisotropic Hardy and Rellich type inequalities for general vector fields // Nonlinear Differential Equations and Applications (NoDEA). – 2019, - V. 26, - No. 13. (Scopus, Web of Science, Q1

5 Sabitbek B. Embedding theorem of Sobolev type spaces on stratified Lie groups // Mathematical Journal. - 2016, - V. 16, - No. 3(61), - P. 166-180. (ККСОН)

6 Sabitbek B., Suragan D. Hardy and Rellich type inequalities on the complex affine group // Eurasian Mathematical Journal. - 2017, - V. 8, -No. 2, - P. 31-39. (ККСОН).

7 Kalmenov T.Sh., Sabitbek B. On Hardy and Rellich type inequalities for the Grushin operator // Mathematical Journal. - 2018, - V. 18, - No. 2(68), - P. 133-142. (ККСОН)

1.2 Homogeneous groups

Let $\mathfrak{g}$ be a Lie algebra which always assumed real and finite dimensional, and $\mathbb{G}$ is the corresponding connected and simply-connected Lie group. The lower central series of $\mathfrak{g}$ is defined inductively by

$$\mathfrak{g}_{(1)} := \mathfrak{g}, \qquad \mathfrak{g}_{(j)} := [\mathfrak{g}, \mathfrak{g}_{(j-1)}].$$

If the lower central series of a Lie algebra $\mathfrak{g}$ terminates at 0 in a finite number of steps then this Lie algebra is called nilpotent. Moreover, if $\mathfrak{g}_{(r+1)} = \{0\}$ and $\mathfrak{g}_{(r)} \neq \{0\}$, then $\mathfrak{g}$ is said to be nilpotent of step r. A Lie algebra $\mathbb{G}$ is nilpotent (of step r) whenever its Lie algebra is nilpotent (of step r). If $\exp: \mathfrak{g} \to \mathbb{G}$

is the exponential map, by the Campbell-Hausdorff formula for $X, Y \in \mathfrak{g}$ sufficiently close to 0 we have

$$\exp X \exp Y = \exp H(X, Y),$$

where $H(X, Y)$ is the Campbell-Hausdorff series which is an infinite linear combination of X and Y and their iterated commutators and H is universal, i.e. independent of $\mathfrak{g}$, and that

$$H(X, Y) = X + Y + \frac{1}{2}[X, Y] + \cdots,$$

where the dots indicate terms of order ≥ 3.

If $\mathfrak{g}$ is nilpotent, the Campbell-Hausdorff series terminates after finitely many terms and defines a polynomial map from $V \times V$ to V, where V is the underlying vector space of $\mathfrak{g}$.

Altogether, we have the following useful properties:

Proposition 1.2.1 [Exponential map and Haar measure] *Let $\mathbb{G}$ be a connected and simply-connected nilpotent Lie group with Lie algebra $\mathfrak{g}$. Then:*

- *The exponential map exp is a dimorphism from $\mathfrak{g}$ to $\mathbb{G}$. Moreover, if $\mathbb{G}$ is identified with $\mathfrak{g}$ via exp, then the group law $(x, y) \mapsto xy$ is a polynomial map.*
- *If μ denotes a Lebesgue measure on $\mathfrak{g}$, then $\mu \circ exp^{-1}$ is a bi-invariant Haar measure on $\mathbb{G}$.*

We refer to [19] for the proof of Proposition 1.2.1.

Definition 1.2.2 [Dilations on a Lie group] *A family of dilations of a Lie algebra $\mathfrak{g}$ is a family of linear mappings*

$$\{\delta_r : r > 0\}$$

from $\mathfrak{g}$ to itself which satisfies:

- *the mappings are of the form*

$$\delta_r = exp(A\, log r\,),$$

where A is a diagonalisable linear operator on $\mathfrak{g}$ with positive eigenvalues.

- *In particular, $\delta_{rs} = \delta_r \delta_s$ for all $r, s > 0$. If $\alpha > 0$ and $\{\delta_r\}$ is a family of dilations on $\mathfrak{g}$, then so is $\{\tilde{\delta}_r\}$, where*

$$\tilde{\delta}_r := \delta_{r^\alpha} = exp(\alpha\, A\, log\, r).$$

By adjusting α we can always assume that the minimum eigenvalue of A is equal to 1.

Let $\mathcal{A}$ be the set of eigenvalues of A and denote by $W_a \subset \mathfrak{g}$ the corresponding eigenfunction space of A, where $a \in \mathcal{A}$. Then we have

$$\delta_r X = r^a X \text{ for } X \in W_a.$$

If $X \in W_a$ and $Y \in W_b$, then

$$\delta_r[X, Y] = [\delta_r X, \delta_r Y] = r^{a+b}[X, Y]$$

and thus $[W_a, W_b] \subset W_{a+b}$. In particular, since $a \geq 1$ for $a \in \mathcal{A}$, we see that $\mathfrak{g}_{(j)} \subset \bigoplus_{a \geq j} W_a$. Since the set $\mathcal{A}$ is finite, it follows that $\mathfrak{g}_{(j)} = \{0\}$ for j sufficiently large. Thus, we obtain:

Proposition 1.2.3 [Lie algebras with dilations are nilpotent] *If a Lie algebra $\mathfrak{g}$ admits a family of dilations then it is nilpotent.*

However, not all nilpotent Lie algebras admit a dilation structure: an example of a (one-dimensional) nilpotent Lie algebra that does not allow any compatible family of dilations was constructed by Dyer [28].

Definition 1.2.4 [Graded Lie algebras and groups] *A Lie algebra $\mathfrak{g}$ is called graded if it is endowed with a vector space decomposition (where all but finitely many of the V_k's are 0)*

$$\mathfrak{g} = \bigoplus_{j=1}^{\infty} V_j \text{ such that } [V_i, V_j] \subset V_{i+j}.$$

Consequently, a Lie group is called graded if it is a connected and simply-connected Lie group whose Lie algebra is graded.

Definition 1.2.5 [Stratified Lie algebras and groups] *A graded Lie algebra $\mathfrak{g}$ is called stratified if V_1 generates $\mathfrak{g}$ as an algebra. In this case, if $\mathfrak{g}$ is nilpotent of step m we have*

$$\mathfrak{g} = \bigoplus_{j=1}^{m} V_j, \ [V_j, V_1] = V_{j+1},$$

and the natural dilations of $\mathfrak{g}$ are given by

$$\delta_r\left(\sum_{k=1}^{m} X_k\right) = \sum_{k=1}^{m} r^k X_k, \ (X_k \in V_k).$$

Consequently, a Lie group is called stratified if it is connected and simply-connected Lie group whose Lie algebra is stratified.

Definition 1.2.6 [Homogeneous groups] *Let δ_r be dilations on $\mathbb{G}$. We say that a Lie group $\mathbb{G}$ is a homogeneous group if:*

- It is a connected and simply-connected nilpotent Lie group $\mathbb{G}$ whose Lie algebra $\mathfrak{g}$ is endowed with a family of dilations $\{\delta_r\}$.

- The maps $exp \circ \delta_r \circ exp^{-1}$ are group automorphism of $\mathbb{G}$.

1.3 Stratified Lie groups

Definition 1.3.1 *A Lie group $\mathbb{G} = (\mathbb{R}^n, \circ)$ is called a stratified (Lie) group if it satisfies the following conditions:*

(a) For some natural numbers $N + N_2 + \cdots + N_r = n$, that is $N = N_1$, the

decomposition $\mathbb{R}^n = \mathbb{R}^N \times ... \times \mathbb{R}^{N_r}$ *is valid, and for every* $\lambda > 0$ *the dilation* $\delta_\lambda: \mathbb{R}^n \to \mathbb{R}^n$ *given by*

$$\delta_\lambda(x) \equiv \delta_\lambda(x', x^{(2)}, \dots, x^{(r)}) := (\lambda x', \lambda^2 x^{(2)}, \dots, \lambda^r x^{(r)})$$

is an automorphism of the group $\mathbb{G}$. *Here* $x' \equiv x^{(1)} \in \mathbb{R}^N$ *and* $x^{(k)} \in \mathbb{R}^{N_k}$ *for* $k = 2, \dots, r$.

(b) Let N *be as in (a) and let* $X_1, \dots, X_N$ *be the left-invariant vector fields on* $\mathbb{G}$ *such that* $X_k(0) = \frac{\partial}{\partial x_k}|_0$ *for* $k = 1, \dots, N$. *Then*

$$rank(Lie\{X_1, \dots, X_N\}) = n,$$

for every $x \in \mathbb{R}^n$, *i.e. the iterated commutators of* $X_1, \dots, X_N$ *span the Lie algebra of* $\mathbb{G}$.

Here, we say that r is called a step of $\mathbb{G}$ and the left-invariant vector fields $X_1, \dots, X_N$ are called the (Jacobian) generators of $\mathbb{G}$. The number

$$Q = \sum_{k=1}^{r} kN_k, \ N_1 = N,$$

is called the homogeneous dimension of a stratified Lie group $\mathbb{G}$. The second order differential operator

$$\mathcal{L} = \sum_{k=1}^{N} X_k^2, \quad N_1 = N, \tag{1.4}$$

is called the (canonical) sub-Laplacian on $\mathbb{G}$. The sub-Laplacian $\mathcal{L}$ is a left-invariant homogeneous hypoelliptic differential operator and it is known that $\mathcal{L}$ is elliptic if and only if the step of $\mathbb{G}$ is equal to 1.

The hypoellipticity of $\mathcal{L}$ means that for a distribution $f \in \mathcal{D}'(\Omega)$ in any open set Ω, if $\mathcal{L}f \in C^\infty(\Omega)$ then $f \in C^\infty(\Omega)$. It is a special case of Hörmander's sum of squares theorem.

The fact that the sub-Laplacian on a stratified Lie group $\mathbb{G}$ has a unique fundamental solution ε was presented by Folland as follows

$$\mathcal{L}\varepsilon = \delta,$$

where δ denotes the Dirac distribution with singularity at the neutral element 0 of $\mathbb{G}$. The fundamental solution $\varepsilon(x, y) = \varepsilon(y^{-1}x)$ is homogeneous of degree $-Q + 2$ and can be written in the form

$$\varepsilon(x, y) = [d(y^{-1}x)]^{2-Q}, \tag{1.5}$$

for some homogeneous d which is called the $\mathcal{L}$-gauge. Thus, the $\mathcal{L}$-gauge is a symmetric homogeneous (quasi-) norm on the stratified group $\mathbb{G} = (\mathbb{R}^n, \circ, \delta_\lambda)$, that is,

- $d(x) > 0$ if and only if $x \neq 0$,
- $d(\delta_\lambda(x)) = \lambda d(x)$ for all $\lambda > 0$ and $x \in \mathbb{G}$,
- $d(x^{-1}) = d(x)$ for all $x \in \mathbb{G}$.

We also recall that the standard Lebesgue measure dx on $\mathbb{R}^n$ is the Haar measure for $\mathbb{G}$ (Proposition 1.6.6 [29]). The left-invariant vector field X_j has an explicit form and satisfies the divergence theorem, see e.g. [29, P. 105-106] for the derivation of the exact formula: more precisely, we can write

$$X_k = \frac{\partial}{\partial x'_k} + \sum_{l=2}^{r} \sum_{m=1}^{N_l} a_{k,m}^{(l)}\left(x', \dots, x^{(l-1)}\right) \frac{\partial}{\partial x_m^{(l)}}. \tag{1.6}$$

We will also use the following notations

$$\nabla_H := (X_1, \dots, X_N)$$

for the horizontal gradient,

$$\mathrm{d}iv_H v := \nabla_H \cdot v$$

for the horizontal divergence,

$$\mathcal{L}_p f := \mathrm{d}iv_H(|\nabla_H f|^{p-2} \nabla_H f), \quad 1 < p < \infty, \tag{1.7}$$

for the horizontal p-Laplacian (or p-sub-Laplacian), and

$$|x'| = \sqrt{x'^2_1 + \cdots + x'^2_N}$$

for the Euclidean norm on $\mathbb{R}^N$.

The explicit representation (1.6) allows us to have the identity

$$\mathrm{d}iv_H\left(\frac{x'}{|x'|^\gamma}\right) = \frac{\sum_{j=1}^{N} |x'|^\gamma X_j x'_j - \sum_{j=1}^{N} x'_j \gamma |x'|^{\gamma-1} X_j |x'|}{|x'|^{2\gamma}} = \frac{N-\gamma}{|x'|^\gamma} \tag{1.8}$$

for all $\gamma \in \mathbb{R}$, $|x'| \neq 0$.

1.3.1 Divergence theorem

Definition 1.3.2 *A bounded open set $\Omega \subset \mathbb{G}$ will be called an admissible domain if its boundary $\partial\Omega$ is piecewise smooth and simple, that is, it has no self-intersections. The condition for the boundary to be simple amounts to $\partial\Omega$ being orientable.*

We now recall the divergence formula in the form of Proposition 3.1 in [30]:

Theorem 1.3.3 *Let $\Omega \subset \mathbb{G}$ be an admissible domain. Let $f_k \in C^1(\Omega) \cap$*

$C(\overline{\Omega}), k = 1, \dots, N$. *Then for each* $k = 1, \dots, N$, *we have*

$$\int_{\Omega} X_k f_k dz = \int_{\partial\Omega} f_k \langle X_k, dz \rangle. \tag{1.9}$$

Consequently, we also have

$$\int_{\Omega} \sum_{k=1}^{N} X_k f_k dz = \int_{\partial\Omega} \sum_{k=1}^{N} f_k \langle X_k, dz \rangle. \tag{1.10}$$

By using the divergence formula, analogues of Green's formulae were obtained in [30, P. 485-487] for general Carnot groups and in [31] for more abstract settings (without the group structure), for another formulation see also [32].

1.3.2 Green's identities for sub-Laplacians

Theorem 1.3.3 [Green's first and second formulae] *Let* $\mathbb{G}$ *be a stratified group and* $\Omega \subset \mathbb{G}$ *be an admissible domain. Then we have the following Green first and second identities:*

- *Let* $v \in C^1(\Omega) \cap C(\overline{\Omega})$ *and* $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$. *Then*

$$\int_{\Omega} \left((\tilde{\nabla} v)u + v\mathcal{L}u\right) dv = \int_{\partial\Omega} v \langle \tilde{\nabla} u, dv \rangle, \tag{1.11}$$

where $\mathcal{L}$ *is the sub-Laplacian on* $\mathbb{G}$ *and where the vector field* $\tilde{\nabla} u$ *is defined by*

$$\tilde{\nabla} u := \sum_{k=1}^{N_1} (X_k u) X_k. \tag{1.12}$$

- *Let* $u, v \in C^2(\Omega) \cap C^1(\overline{\Omega})$. *Then*

$$\int_{\Omega} (u\mathcal{L}v - v\mathcal{L}u) dv = \int_{\partial\Omega} (u\langle \tilde{\nabla}, dv \rangle - v \langle \tilde{\nabla} u, dv \rangle). \tag{1.13}$$

1.3.3 Engel group

A well-known stratified group with step three is the Engel group, which can be denoted by $\mathbb{E}$. Topologically $\mathbb{E}$ is $\mathbb{R}^4$ with the group law of $\mathbb{E}$, which is given by

$$x \circ y = (x_1 + y_1, x_2 + y_2, x_3 + y_3 + P_1, x_4 + y_4 + P_2),$$

where

$$P_1 = \frac{1}{2}(x_1 y_2 - x_2 y_1),$$
$$P_2 = \frac{1}{2}(x_1 y_3 - x_3 y_1) + \frac{1}{12}(x_1^2 y_2 - x_1 y_1 (x_2 + y_2) + x_2 y_1^2).$$

The left-invariant vector fields of $\mathbb{E}$ are generated by the basis

$$
\begin{aligned}
X_1 &= \frac{\partial}{\partial x_1} - \frac{x_2}{2}\frac{\partial}{\partial x_3} - \left(\frac{x_3}{2} - \frac{x_1 x_2}{12}\right)\frac{\partial}{\partial x_4},\\
X_2 &= \frac{\partial}{\partial x_2} + \frac{x_1}{2}\frac{\partial}{\partial x_3} + \frac{x_1^2}{12}\frac{\partial}{\partial x_4},\\
X_3 &= \frac{\partial}{\partial x_3} + \frac{x_1}{2}\frac{\partial}{\partial x_4},\\
X_4 &= \frac{\partial}{\partial x_4}.
\end{aligned}
$$

1.3.4 Heisenberg group

Let us give a brief introduction of the Heisenberg group. Let $\mathbb{H}^n$ be the Heisenberg group, that is, the set $\mathbb{R}^{2n+1}$ equipped with the group law

$$\xi \circ \tilde{\xi} := \left(x + \tilde{x}, y + \tilde{y}, t + \tilde{t} + 2\sum_{i=1}^{n} (\tilde{x}_i y_i - x_i \tilde{y}_i)\right),$$

where $\xi := (x, y, t) \in \mathbb{H}^n$, $x := (x_1, \dots, x_n)$, $y := (y_1, \dots, y_n)$, and $\xi^{-1} = -\xi$ is the inverse element of ξ with respect to the group law. The dilation operation of the Heisenberg group with respect to the group law has the following form

$$\delta_\lambda(\xi) := (\lambda x, \lambda y, \lambda^2 t) \text{ for } \lambda > 0.$$

The Lie algebra $\mathfrak{h}$ of the left-invariant vector fields on the Heisenberg group $\mathbb{H}^n$ is spanned by

$$X_i := \frac{\partial}{\partial x_i} + 2y_i \frac{\partial}{\partial t} \text{ for } 1 \le i \le n,$$

$$Y_i := \frac{\partial}{\partial y_i} - 2x_i \frac{\partial}{\partial t} \text{ for } 1 \le i \le n,$$

and with their (non-zero) commutator

$$[X_i, Y_i] = -4\frac{\partial}{\partial t}.$$

The horizontal gradient of $\mathbb{H}^n$ is given by

$$\nabla_H := (X_1, \dots, X_n, Y_1, \dots, Y_n),$$

so the sub-Laplacian on $\mathbb{H}^n$ is given by

$$\mathcal{L} := \sum_{i=1}^{n} \left(X_i^2 + Y_i^2\right).$$

1.3.5 Quaternion Heisenberg group

Let $\mathbb{H}$ be the set of all quaternions $x = x_0 + x_1 i_1 + x_2 i_2 + x_3 i_3$, where $x_0, x_1, x_2, x_3 \in \mathbb{R}$ and $1, i_1, i_2, i_3$ are the basis elements of $\mathbb{H}$ with following rules of multiplication

$$i_1^2 = i_2^2 = i_3^2 = i_1 i_2 i_3 = -1, \qquad i_1 i_2 = -i_2 i_1 = i_3, \qquad i_2 i_3 = -i_3 i_2 = i_1,$$
$$i_3 i_1 = -i_1 i_3 = i_2.$$

Let $x = x_0 + x_1 i_1 + x_2 i_2 + x_3 i_3 \in \mathbb{H}$. Then the real part of x is the real number x_0 and the imaginary part of x is the point $(x_1, x_2, x_3) \in \mathbb{R}^3$. Also, the real and imaginary parts of x are denoted by $\Re x$ and $\Im x$, respectively. It will be useful further to denote the imaginary parts such as

$$\Im_1 x = x_1, \quad \Im_2 x = x_2, \quad \Im_3 x = x_3.$$

The conjugate of x is denoted by

$$\overline{x} = x_0 - x_1 i_1 - x_2 i_2 - x_3 i_3,$$

and the modulus $|x|$ is defined by

$$|x|^2 = x\overline{x} = \sum_{j=0}^{3} x_j^2.$$

The Grassmannian product (or the quaternion product) of x and y is defined by

$$xy = (x_0 y_0 - \Im x \cdot \Im y) + (x_0 \Im y + y_0 \Im x + \Im x \times \Im y),$$

where

$$\Im x \times \Im y = \det \begin{pmatrix} i_1 & i_2 & i_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{pmatrix}.$$

Let $\mathbb{H}_q = \mathbb{H} \times \mathbb{R}^3$. Then $\mathbb{H}_q$ becomes a non-commutative (Lie) group with the group law

$$\begin{aligned}(x, t_1, t_2, t_3) &\circ (y, \tau_1, \tau_2, \tau_3) \\ &= (x + y, t_1 + \tau_1 - 2\Im_1(\overline{y}x), t_2 + y_2 - 2\Im_2(\overline{y}x), t_3 + \tau_3 - 2\Im_3(\overline{y}x)),\end{aligned}$$

for all $(x, t), (y, \tau) \in \mathbb{H}_q$. We note that $e = (0,0,0,0)$ is the identity element of $\mathbb{H}_q$ and the inverse of an element $(x, t_1, t_2, t_3) \in \mathbb{H}_q$ is $(-x, -t_1, -t_2, -t_3)$. The Haar measure on coincides with the Lebesgue measure on $\mathbb{H} \times \mathbb{R}^3$ which is denoted by $dxdt$. Let $\mathfrak{h}_q$ be the Lie algebra of left invariant vector fields on $\mathbb{H}_q$. A basis of $\mathfrak{h}_q$ is given by $\{X_0, X_1, X_2, X_3\}$ and $\{T_1, T_2, T_3\}$, where

$$\begin{aligned}
X_0 &= \frac{\partial}{\partial x_0} - 2x_1 \frac{\partial}{\partial t_1} - 2x_2 \frac{\partial}{\partial t_2} - 2x_3 \frac{\partial}{\partial t_3}, \\
X_1 &= \frac{\partial}{\partial x_1} + 2x_0 \frac{\partial}{\partial t_1} - 2x_3 \frac{\partial}{\partial t_2} + 2x_2 \frac{\partial}{\partial t_3}, \\
X_2 &= \frac{\partial}{\partial x_2} + 2x_3 \frac{\partial}{\partial t_1} + 2x_0 \frac{\partial}{\partial t_2} - 2x_1 \frac{\partial}{\partial t_3},
\end{aligned}$$

$$X_3 = \frac{\partial}{\partial x_3} - 2x_2 \frac{\partial}{\partial t_1} + 2x_1 \frac{\partial}{\partial t_2} + 2x_0 \frac{\partial}{\partial t_3},$$

and

$$T_k = \frac{\partial}{\partial t_k}, \quad k = 1,2,3.$$

The Lie brackets of these vector fields are given by

$$\begin{aligned}
&[X_0, X_1] = [X_3, X_2] = 4T_1,\\
&= [X_1, X_3] = 4T_2,\\
&= [X_2, X_1] = 4T_3.
\end{aligned}$$

Thus, the sub-Laplacian on $\mathbb{H}_q$ is given by

$$= \sum_{j=0}^{3} X_j^2 = \Delta_x - 4|x|^2 \Delta_t - 4 \sum_{k=1}^{3} (i_k x \cdot \nabla_x) \frac{\partial}{\partial t_k}, \tag{1.14}$$

where

$$\Delta_x = \sum_{k=0}^{3} \frac{\partial^2}{\partial x_k^2}, \quad \text{and} \quad \Delta_t = \sum_{k=1}^{3} \frac{\partial^2}{\partial t_k^2}.$$

Note that the fundamental solution of the sub-Laplacian on $\mathbb{H}_q$ was found by Tie and Wong in [33]. We restate their results in the following theorem.

Theorem 1.3.5 *The fundamental solution $\Gamma(\xi)$ of the sub-Laplacian on the quaternion Heisenberg group $\mathbb{H}_q$ is given by*

$$\Gamma(\xi) := \Gamma(|x|, t) = \frac{2}{(2\pi)^{7/2}|x|^2} \int_{S^2} \frac{1}{(|x|^2 - i(t \cdot n))^3} d\sigma, \tag{1.15}$$

where $\xi = (x, t) \in \mathbb{H}_q$, $n = (n_1, n_2, n_3)$ is a point on the unit sphere S^2 in $\mathbb{R}^3$ with centre at the origin, and $d\sigma$ is the surface measure on S^2. That is,

$$\mathcal{L}\Gamma_\zeta = -\delta_\zeta, \tag{1.16}$$

where $\Gamma_\zeta(\xi) = \Gamma(\zeta^{-1} \circ \xi)$ and δ_ζ is the Dirac distribution at $\zeta \equiv (y, \tau) \in \mathbb{H}_q$.

2 GEOMETRIC HARDY AND HARDY-SOBOLEV TYPE INEQUALITIES

In the Euclidean setting, a geometric Hardy inequality in a (Euclidean) convex domain Ω has the following form

$$\int_{\Omega} |\nabla u|^2 dx \geq \frac{1}{4}\int_{\Omega} \frac{|u|^2}{dist(x,\partial\Omega)^2} dx, \tag{2.1}$$

for $u \in C_0^{\infty}(\Omega)$ with the sharp constant $1/4$. There is a number of studies related to this subject [34-39].

The Hardy inequality in the half-space on the Heisenberg group was shown by Luan and Young [40] in the form

$$\int_{\mathbb{H}^+} |\nabla_H u|^2 d\xi \geq \frac{1}{4}\int_{\mathbb{H}^+} \frac{|x|^2+|y|^2}{t^2} |u|^2 d\xi. \tag{2.2}$$

An alternative proof of this inequality was given by Simon Larson, where the author generalised it to any half-space of the Heisenberg group,

$$\int_{\mathbb{H}^+} |\nabla_H u|^2 d\xi \geq \frac{1}{4}\int_{\mathbb{H}^+} \frac{\sum_{i=1}^{n} \langle X_i(\xi),\nu\rangle^2 + \langle Y_i(\xi),\nu\rangle^2}{dist(\xi,\partial\mathbb{H}^+)^2} |u|^2 d\xi,$$

where X_i and Y_i (for $i = 1, \dots, n$) are left-invariant vector fields on the Heisenberg group, ν is the Riemannian outer unit normal [41] to the boundary. Also, there is the L^p-generalisation of the above inequality

$$\int_{\mathbb{H}^+} |\nabla_H u|^p d\xi \geq \left(\frac{p-1}{p}\right)^p \int_{\mathbb{H}^+} \frac{\sum_{i=1}^{n} |\langle X_i(\xi),\nu\rangle|^p + |\langle Y_i(\xi),\nu\rangle|^p}{dist(\xi,\partial\mathbb{H}^+)^p} |u|^p d\xi.$$

This Chapter is devoted to present the geometric Hardy and Hardy-Sobolev inequalities on the stratified Lie group and the Heisenberg group, respectively.

The main results of the chapter are as follow:

Geometric L^2-Hardy inequality on $\mathbb{G}^+$. Let $\mathbb{G}^+$ be a half-space of a stratified group $\mathbb{G}$. Then for all $\beta \in \mathbb{R}$ we have

$$\int_{\mathbb{G}^+} |\nabla_{\mathbb{G}} u|^2 d \geq -(\beta^2+\beta)\int_{\mathbb{G}^+} \frac{\sum_{i=1}^{N} \langle X_i(x),\nu\rangle^2}{dist(x,\partial\mathbb{G}^+)^2} |u|^2 dx$$
$$+\beta\int_{\mathbb{G}^+}\sum_{i=1}^{N} \frac{X_i\langle X_i(x),\nu\rangle}{dist(x,\partial\mathbb{G}^+)} |u|^2 dx,$$

for all $u \in C_0^{\infty}(\mathbb{G}^+)$.

Geometric L^p-Hardy type inequality on $\mathbb{G}^+$. Let $\mathbb{G}^+$ be a half-space of a stratified group $\mathbb{G}$. Then for all $\beta \in \mathbb{R}$ we have

$$\int_{\mathbb{G}^+} \sum_{i=1}^{N} |X_i u|^p dx \geq -(p-1)(|\beta|^{\frac{p}{p-1}} + \beta) \int_{\mathbb{G}^+} \frac{\sum_{i=1}^{N} |\langle X_i(x), \nu \rangle|^p}{dist(x, \partial \mathbb{G}^+)^p} |u|^p dx$$
$$+\beta(p-1) \int_{\mathbb{G}^+} \sum_{i=1}^{N} \left(\frac{|\langle X_i(x), \nu \rangle|}{dist(x, \partial \mathbb{G}^+)} \right)^{p-2} \frac{X_i \langle X_i(x), \nu \rangle}{dist(x, \partial \mathbb{G}^+)} |u|^p dx,$$

for all $u \in C_0^\infty(\mathbb{G}^+)$, $1 < p < \infty$.

Geometric L^p-Hardy inequality on $\mathbb{G}^+$. Let $\mathbb{G}^+ := \{x \in \mathbb{G}: \langle x, \nu \rangle > d\}$ be a half-space of a stratified group $\mathbb{G}$. Then for all $u \in C_0^\infty(\mathbb{G})$, $\beta \in \mathbb{R}$ and $p > 1$ we have

$$\int_{\mathbb{G}^+} |\nabla_{\mathbb{G}} u|^p dx \geq -(p-1)(|\beta|^{\frac{p}{p-1}} + \beta) \int_{\mathbb{G}^+} \frac{(\sum_{i=1}^{N} \langle X_i(x), \nu \rangle^2)^{p/2}}{dist(x, \partial \mathbb{G}^+)^p} |u|^p dx$$
$$+\beta \int_{\mathbb{G}^+} \frac{{}_p(dist(x, \partial \mathbb{G}^+))}{dist(x, \partial \mathbb{G}^+)^{p-1}} |u|^p dx.$$

Geometric L^2-Hardy inequality on a convex domain of $\mathbb{G}$. Let Ω be a convex domain of a stratified group $\mathbb{G}$. Then for $\beta < 0$ we have

$$\int_{\Omega} |\nabla_{\mathbb{G}} u|^2 dx \geq -(\beta^2 + \beta) \int_{\Omega} \frac{\sum_{i=1}^{N} \langle X_i(x), \nu \rangle^2}{dist(x, \partial \Omega)^2} |u|^2 dx$$
$$+ \beta \int_{\Omega} \sum_{i=1}^{N} \frac{X_i \langle X_i(x), \nu \rangle}{dist(x, \partial \Omega)} |u|^2 dx$$

for all $u \in C_0^\infty(\Omega)$.

Geometric L^p-Hardy type inequality on a convex domain of $\mathbb{G}$. Let Ω be a convex domain of a stratified group $\mathbb{G}$. Then for $\beta < 0$ we have

$$\int_{\Omega} \sum_{i=1}^{N} |X_i u|^p dx \geq -(p-1)(|\beta|^{\frac{p}{p-1}} + \beta) \int_{\Omega} \frac{\sum_{i=1}^{N} |\langle X_i(x), \nu \rangle|^p}{dist(x, \partial \Omega)^p} |u|^p dx$$
$$+\beta(p-1) \int_{\Omega} \sum_{i=1}^{N} \left(\frac{|\langle X_i(x), \nu \rangle|}{dist(x, \partial \Omega)} \right)^{p-2} \left(\frac{X_i \langle X_i(x), \nu \rangle}{dist(x, \partial \Omega)} \right) |u|^p dx,$$

for all $u \in C_0^\infty(\Omega)$.

Geometric L^p-Hardy inequality on $\mathbb{H}^+$. Let $\mathbb{H}^+ := \{\xi \in \mathbb{H}^n: \langle \xi, \nu \rangle > d\}$ be a half-space of the Heisenberg group. Then for all $u \in C_0^\infty(\mathbb{H}^+)$ and $p > 1$ we have

$$\int_{\mathbb{H}^+} |\nabla_H u|^p d\xi \geq \left(\frac{p-1}{p}\right)^p \int_{\mathbb{H}^+} \frac{\mathcal{W}(\xi)^p}{dist(\xi, \partial\mathbb{H}^+)^p} |u|^p d\xi,$$

where $\mathcal{W}(\xi) := (\sum_{i=1}^n \langle X_i(\xi), \nu\rangle^2 + \langle Y_i(\xi), \nu\rangle^2)^{1/2}$ and the constant is sharp.

Geometric Hardy-Sobolev inequality on $\mathbb{H}^+$. Let $\mathbb{H}^+ := \{\xi \in \mathbb{H}^n : \langle \xi, \nu\rangle > d\}$ be a half-space of the Heisenberg group. Then for all $u \in C_0^\infty(\mathbb{H}^+)$ and $2 \leq p < Q$ with $Q = 2n+1$, there exists some $C > 0$ such that we have

$$\left(\int_{\mathbb{H}^+} |\nabla_H u|^p d\xi - \left(\frac{p-1}{p}\right)^p \int_{\mathbb{H}^+} \frac{\mathcal{W}(\xi)^p}{dist(\xi, \partial\mathbb{H}^+)^p} |u|^p d\xi\right)^{\frac{1}{p}} \geq C \left(\int_{\mathbb{H}^+} |u|^{p^*} d\xi\right)^{\frac{1}{p^*}},$$

where $dist(\xi, \partial\mathbb{H}^+) := \langle \xi, \nu\rangle - d$ is the distance from ξ to the boundary and $p^* := Qp/(Q-p)$.

Geometric L^p-Hardy inequality on starshaped sets of $\mathbb{G}$. Let Ω be a starshaped set on a Carnot group. Then for every $\gamma \in \mathbb{R}$ and $p > 1$ we have the following Hardy inequality

$$\int_\Omega |\nabla_H f(x)|^p dx \geq -(p-1)(|\gamma|^{\frac{p}{p-1}} + \gamma) \int_\Omega \frac{|\nabla_H \langle Z(x), n(x)\rangle|^p}{|\langle Z(x), n(x)\rangle|^p} |f(x)|^p dx$$
$$+\gamma \int_\Omega \frac{\mathcal{L}_p(\langle Z(x), n(x)\rangle)}{|\langle Z(x), n(x)\rangle|^{p-1}} |f(x)|^p dx,$$

for every function $f \in C_0^\infty(\Omega)$.

2.1 Geometric L^2-Hardy inequality on half-spaces

In this section we present the geometric L^2-Hardy inequality on the half-space of $\mathbb{G}$. We define the half-space as follow

$$\mathbb{G}^+ := \{x \in \mathbb{G} : \langle x, \nu\rangle > d\},$$

where $\nu := (\nu_1, \dots, \nu_r)$ with $\nu_j \in \mathbb{R}^{N_j}$, $j = 1, \dots, r$, is the Riemannian outer unit normal to $\partial\mathbb{G}^+$ (see [41]) and $d \in \mathbb{R}$. The Euclidean distance to the boundary $\partial\mathbb{G}^+$ is denoted by $dist(x, \partial\mathbb{G}^+)$ and defined as follows

$$dist(x, \partial\mathbb{G}^+) = \langle x, \nu\rangle - d. \tag{2.3}$$

Moreover, there is an angle function on $\partial\mathbb{G}^+$ which is defined by Garofalo as

$$\mathcal{W}(x) = \sqrt{\sum_{i=1}^N \langle X_i(x), \nu\rangle^2}. \tag{2.4}$$

Theorem 2.1.1 *Let* $\mathbb{G}^+$ *be a half-space of a stratified group* $\mathbb{G}$. *Then for all* $\beta \in \mathbb{R}$ *we have*

$$\int_{\mathbb{G}^+} |\nabla_{\mathbb{G}} u|^2 dx \geq C_1(\beta) \int_{\mathbb{G}^+} \frac{\mathcal{W}(x)^2}{dist(x,\partial\mathbb{G}^+)^2} |u|^2 dx + \beta \int_{\mathbb{G}^+} \sum_{i=1}^{N} \frac{X_i \langle X_i(x), \nu\rangle}{dist(x,\partial\mathbb{G}^+)} |u|^2 dx, \tag{2.5}$$

for all $u \in C_0^\infty(\mathbb{G}^+)$ *and where* $C_1(\beta) := -(\beta^2 + \beta)$.

Remark 2.1.2 If $\mathbb{G}$ has step $r = 2$, then for $i = 1, \dots, N$ we have the following left-invariant vector fields

$$X_i = \frac{\partial}{\partial x'_i} + \sum_{s=1}^{N_2} \sum_{m=1}^{N} a_{m,i}^s x'_m \frac{\partial}{\partial x''_s}, \tag{2.6}$$

where $a_{m,i}^s$ are the group constants (see, e.g. [42, Formula (2.14)] for the definition). Also we have $x := (x', x'')$ with $x' = (x'_1, \dots, x_{N'})$, $x'' = (x''_1, \dots, x''_{N_2})$, and also $\nu := (\nu', \nu'')$ with $\nu' = (\nu'_1, \dots, \nu_{N'})$ and $\nu'' = (\nu''_1, \dots, \nu''_{N_2})$.

Corollary 2.1.3 *Let* $\mathbb{G}^+$ *be a half-space of a stratified group* $\mathbb{G}$ *of step* $r = 2$. *For all* $\beta \in \mathbb{R}$ *and* $u \in C_0^\infty(\mathbb{G}^+)$ *we have*

$$\int_{\mathbb{G}^+} |\nabla_{\mathbb{G}} u|^2 dx \geq C_1(\beta) \int_{\mathbb{G}^+} \frac{\mathcal{W}(x)^2}{dist(x,\partial\mathbb{G}^+)^2} |u|^2 dx + K(a,\nu,\beta) \int_{\mathbb{G}^+} \frac{|u|^2}{dist(x,\partial\mathbb{G}^+)} dx, \tag{2.7}$$

where $C_1(\beta) := -(\beta^2 + \beta)$ *and* $K(a,\nu,\beta) := \beta \sum_{s=1}^{N_2} \sum_{i=1}^{N} a_{i,i}^s \nu_s''$.

Proof of Theorem 2.1.1. To prove inequality (2.5) we use the method of factorization. Thus, for any $W := (W_1, \dots, W_N)$, $W_i \in C^1(\mathbb{G}^+)$ real-valued, which will be chosen later, by a simple computation we have

$$\begin{aligned}
0 &\leq \int_{\mathbb{G}^+} |\nabla_{\mathbb{G}} u + \beta W u|^2 dx \\
&= \int_{\mathbb{G}^+} |(X_1 u, \dots, X_N u) + \beta(W_1, \dots, W_N) u|^2 dx \\
&= \int_{\mathbb{G}^+} |(X_1 u + \beta W_1 u, \dots, X_N u + \beta W_N u)|^2 dx \\
&= \int_{\mathbb{G}^+} \sum_{i=1}^{N} |X_i u + \beta W_i u|^2 dx \\
&= \int_{\mathbb{G}^+} \sum_{i=1}^{N} [|X_i u|^2 + 2\mathrm{Re}\beta W_i u X_i u + \beta^2 W_i^2 |u|^2] dx \\
&= \int_{\mathbb{G}^+} \sum_{i=1}^{N} [|X_i u|^2 + \beta W_i X_i |u|^2 + \beta^2 W_i^2 |u|^2] dx \\
&= \int_{\mathbb{G}^+} \sum_{i=1}^{N} [|X_i u|^2 - \beta (X_i W_i) |u|^2 + \beta^2 W_i^2 |u|^2] dx.
\end{aligned}$$

From the above expression we get the inequality

$$\int_{\mathbb{G}^+} |\nabla_{\mathbb{G}} u|^2 dx \geq \int_{\mathbb{G}^+} \sum_{i=1}^{N} [(\beta(X_i W_i) - \beta^2 W_i^2)|u|^2] dx. \tag{2.8}$$

Let us now take W_i as

$$W_i(x) = \frac{\langle X_i(x),\nu\rangle}{dist(x,\partial\mathbb{G}^+)} = \frac{\langle X_i(x),\nu\rangle}{\langle x,\nu\rangle - d}, \tag{2.9}$$

where

$$X_i(x) = (\overbrace{(0,\dots,1}^{i} \dots,0, a_{i,1}^{(2)}(x'),\dots, a_{i,N_r}^{(r)}(x',x^{(2)},\dots,x^{(r-1)})),$$

and

$$\nu = (\nu_1,\nu_2,\dots,\nu_r),\ \nu_j \in \mathbb{R}^{N_j}.$$

Now $W_i(x)$ can be written as

$$W_i(x) = \frac{\nu_{1,i} + \sum_{l=2}^{r}\sum_{m=1}^{N_l} a_{i,m}^{(l)}(x',\dots,x^{(l-1)})\nu_{l,m}}{\sum_{l=1}^{r} x^{(l)}\cdot\nu_l - d}.$$

By a direct computation we have

$$\begin{aligned} X_iW_i(x) &= \frac{X_i\langle X_i(x),\nu\rangle dist(x,\partial\mathbb{G}^+) - \langle X_i(x),\nu\rangle X_i(dist(x,\partial\mathbb{G}^+))}{dist(x,\partial\mathbb{G}^+)^2} \\ &= \frac{X_i\langle X_i(x),\nu\rangle}{dist(x,\partial\mathbb{G}^+)} - \frac{\langle X_i(x),\nu\rangle^2}{dist(x,\partial\mathbb{G}^+)^2}, \end{aligned} \tag{2.10}$$

where

$$\begin{aligned} X_i(dist(x,\partial\mathbb{G}^+)) &= X_i\left(\sum_{k=1}^{N} x'_k\nu_{1,k} + \sum_{l=2}^{r}\sum_{m=1}^{N_l} x_m^{(l)}\nu_{l,m} - d\right) \\ &= \nu_{1,i} + \sum_{l=2}^{r}\sum_{m=1}^{N_l} a_{i,m}^{(l)}(x',\dots,x^{(l-1)})\nu_{l,m} \\ &= \langle X_i(x),\nu\rangle. \end{aligned}$$

Inserting the expression (2.10) in (2.8) we get

$$\int_{\mathbb{G}^+} |\nabla_{\mathbb{G}} u|^2 dx \geq -(\beta^2+\beta)\int_{\mathbb{G}^+}\sum_{i=1}^{N}\frac{\langle X_i(x),\nu\rangle^2}{dist(x,\partial\mathbb{G}^+)^2}|u|^2dx$$
$$+\beta\int_{\mathbb{G}^+}\sum_{i=1}^{N}\frac{X_i\langle X_i(x),\nu\rangle}{dist(x,\partial\mathbb{G}^+)}|u|^2dx.$$

The proof of Theorem 2.1.1 is finished.

As consequences of Theorem 2.1.1, we have the geometric Hardy inequalities on the half-space without an angle function, which seems an interesting new result on

$\mathbb{G}$.

Corollary 2.1.4 *Let* $\mathbb{G}^+$ *be a half-space of a stratified group* $\mathbb{G}$. *Then we have*

$$\int_{\mathbb{G}^+} |\nabla_{\mathbb{G}} u|^2 dx \geq \frac{1}{4} \int_{\mathbb{G}^+} \frac{|u|^2}{dist(x, \partial\mathbb{G}^+)^2} dx, \tag{2.11}$$

for all $u \in C_0^\infty(\mathbb{G}^+)$.

Proof of Corollary 2.1.4. Let $x := (x', x^{(2)}, \ldots, x^{(r)}) \in \mathbb{G}$ with $x' = (x'_1, \ldots, x'_N)$ and $x^{(j)} \in \mathbb{R}^{N_j}$, $j = 2, \ldots, r$. By taking $\nu := (\nu', 0, \ldots, 0)$ with $\nu' = (\nu'_1, \ldots, \nu'_N)$, we have that

$$X_i(x) = (\overbrace{(0, \ldots, 1}^{i} \ldots, 0, a_{i,1}^{(2)}(x'), \ldots, a_{i,N_r}^{(r)}(x', x^{(2)}, \ldots, x^{(r-1)})),$$

we have

$$\sum_{i=1}^{N} \langle X_i(x), \nu \rangle^2 = \sum_{i=1}^{N} (\nu_{i'})^2 = |\nu'|^2 = 1,$$

and

$$X_i \langle X_i(x), \nu \rangle = X_i \nu'_i = 0.$$

Inserting the above expressions in inequality (2.5) we arrive at

$$\int_{\mathbb{G}^+} |\nabla_{\mathbb{G}} u|^2 dx \geq -(\beta^2 + \beta) \int_{\mathbb{G}^+} \frac{|u|^2}{dist(x, \partial\mathbb{G}^+)^2} dx.$$

For optimisation we differentiate the right-hand side of integral with respect to β, then we have

$$-2\beta - 1 = 0,$$

which implies

$$\beta = -\frac{1}{2}.$$

We complete the proof.

We also have the geometric uncertainty principle on the half-space of $\mathbb{G}^+$.

Corollary 2.1.5 *Let* $\mathbb{G}^+$ *be a half-space of a stratified group* $\mathbb{G}$. *Then we have*

$$\left(\int_{\mathbb{G}^+} |\nabla_{\mathbb{G}} u|^2 dx\right)^{\frac{1}{2}} \left(\int_{\mathbb{G}^+} dist(x, \partial\mathbb{G}^+)^2 |u|^2 dx\right)^{\frac{1}{2}} \geq \frac{1}{2} \int_{\mathbb{G}^+} |u|^2 dx, \tag{2.12}$$

for all $u \in C_0^\infty(\mathbb{G}^+)$.

P r o o f o f C o r o l l a r y 2.1.5. By using (2.11) and the Cauchy-Schwarz inequality we get

$$\int_{\mathbb{G}^+} |\nabla_{\mathbb{G}} u|^2 dx \int_{\mathbb{G}^+} dist(x, \partial\mathbb{G}^+)^2 |u|^2 dx$$
$$\geq \frac{1}{4} \int_{\mathbb{G}^+} \frac{1}{dist(x, \partial\mathbb{G}^+)^2} |u|^2 dx \int_{\mathbb{G}^+} dist(x, \partial\mathbb{G}^+)^2 |u|^2 dx$$
$$\geq \frac{1}{4} \left(\int_{\mathbb{G}^+} |u|^2 dx \right)^2.$$

To demonstrate our general result in a particular case, here we consider the Heisenberg group, which is a well-known example of step $r = 2$ (stratified) group.

C o r o l l a r y 2.1.6 *Let* $\mathbb{H}^+ = \{(x_1, x_2, x_3) \in \mathbb{H} \mid x_3 > 0\}$ *be a half-space of the Heisenberg group* $\mathbb{H}$. *Then for any* $u \in C_0^\infty(\mathbb{H}^+)$ *we have*

$$\int_{\mathbb{H}^+} |\nabla_{\mathbb{H}} u|^2 dx \geq \int_{\mathbb{H}^+} \frac{|x_1|^2 + |x_2|^2}{x_3^2} |u|^2 dx, \tag{2.13}$$

where $\nabla_H = \{X_1, X_2\}$.

P r o o f o f C o r o l l a r y 2.1.6. Recall that the left-invariant vector fields on the Heisenberg group are generated by the basis

$$X_1 = \frac{\partial}{\partial x_1} + 2x_2 \frac{\partial}{\partial x_3},$$
$$X_2 = \frac{\partial}{\partial x_2} - 2x_1 \frac{\partial}{\partial x_3},$$

with the commutator

$$[X_1, X_2] = -4 \frac{\partial}{\partial x_3}.$$

For $x = (x_1, x_2, x_3)$, choosing $\nu = (0,0,1)$ as the unit vector in the direction of x_3 and taking $d = 0$ in inequality (2.5), we get

$$X_1(x) = (1,0,2x_2) \text{ and } X_2(x) = (0,1,-2x_1),$$

and

$$\langle X_1(x), \nu \rangle = 2x_2, \quad \text{and} \quad \langle X_2(x), \nu \rangle = -2x_1,$$

$$X_1 \langle X_1(x), \nu \rangle = 0, \quad \text{and} \quad X_2 \langle X_2(x), \nu \rangle = 0.$$

Therefore, with $\mathcal{W}(x)$ as in (2.4), we have

$$\frac{\mathcal{W}(x)^2}{dist(x,\partial\mathbb{G}^+)^2} = 4\frac{|x_1|^2+|x_2|^2}{x_3^2}.$$

Substituting these into inequality (2.5) we arrive at

$$\int_{\mathbb{H}^+} |\nabla_H u|^2 dx \geq \int_{\mathbb{H}^+} \frac{|x_1|^2+|x_2|^2}{x_3^2}|u|^2 dx,$$

taking $\beta = -\frac{1}{2}$.

Let us present an example for the step $r = 3$ (stratified) groups. A well-known stratified group with step three is the Engel group, which can be denoted by $\mathbb{E}$. Topologically $\mathbb{E}$ is $\mathbb{R}^4$ with the group law of $\mathbb{E}$, which is given by

$$x \circ y = (x_1 + y_1, x_2 + y_2, x_3 + y_3 + P_1, x_4 + y_4 + P_2),$$

where

$$\begin{aligned}
P_1 &= \frac{1}{2}(x_1 y_2 - x_2 y_1),\\
P_2 &= \frac{1}{2}(x_1 y_3 - x_3 y_1) + \frac{1}{12}(x_1^2 y_2 - x_1 y_1(x_2 + y_2) + x_2 y_1^2).
\end{aligned}$$

The left-invariant vector fields of $\mathbb{E}$ are generated by the basis

$$\begin{aligned}
X_1 &= \frac{\partial}{\partial x_1} - \frac{x_2}{2}\frac{\partial}{\partial x_3} - \left(\frac{x_3}{2} - \frac{x_1 x_2}{12}\right)\frac{\partial}{\partial x_4},\\
X_2 &= \frac{\partial}{\partial x_2} + \frac{x_1}{2}\frac{\partial}{\partial x_3} + \frac{x_1^2}{12}\frac{\partial}{\partial x_4},\\
X_3 &= \frac{\partial}{\partial x_3} + \frac{x_1}{2}\frac{\partial}{\partial x_4},\\
X_4 &= \frac{\partial}{\partial x_4}.
\end{aligned}$$

Corollary 2.1.7 *Let* $\mathbb{E}^+ = \{x := (x_1, x_2, x_3, x_4) \in \mathbb{E} \mid \langle x, \nu \rangle > 0\}$ *be a half-space of the Engel group* $\mathbb{E}$. *Then for all* $\beta \in \mathbb{R}$ *and* $u \in C_0^\infty(\mathbb{E}^+)$ *we have*

$$\begin{aligned}
\int_{\mathbb{E}^+} |\nabla_{\mathbb{E}} u|^2 dx &\geq C_1(\beta)\int_{\mathbb{E}^+} \frac{\langle X_1(x),\nu\rangle^2 + \langle X_2(x),\nu\rangle^2}{dist(x,\partial\mathbb{E}^+)^2}|u|^2 dx \\
&+ \frac{\beta}{3}\int_{\mathbb{E}^+} \frac{x_2\nu_4}{dist(x,\partial\mathbb{E}^+)}|u|^2 dx,
\end{aligned} \tag{2.14}$$

where $\nabla_{\mathbb{E}} = \{X_1, X_2\}$, $\nu := (\nu_1, \nu_2, \nu_3, \nu_4)$, *and* $C_1(\beta) = -(\beta^2 + \beta)$.

Remark 2.1.8 If we take $\nu_4 = 0$ in (2.14), then we have the following inequality on $\mathbb{E}$, by taking $\beta = -\frac{1}{2}$,

$$\int_{\mathbb{E}^+} |\nabla_{\mathbb{E}} u|^2 dx \geq \frac{1}{4} \int_{\mathbb{E}^+} \frac{\langle X_1(x), \nu\rangle^2 + \langle X_2(x), \nu\rangle^2}{dist(x, \partial\mathbb{E}^+)^2} |u|^2 dx.$$

Proof of Corollary 2.1.7. As we mentioned, the Engel group has the following basis of the left-invariant vector fields

$$X_1 = \frac{\partial}{\partial x_1} - \frac{x_2}{2}\frac{\partial}{\partial x_3} - \left(\frac{x_3}{2} - \frac{x_1 x_2}{12}\right)\frac{\partial}{\partial x_4},$$
$$X_2 = \frac{\partial}{\partial x_2} + \frac{x_1}{2}\frac{\partial}{\partial x_3} + \frac{x_1^2}{12}\frac{\partial}{\partial x_4},$$

with the following two (non-zero) commutators

$$X_3 = [X_1, X_2] = \frac{\partial}{\partial x_3} + \frac{x_1}{2}\frac{\partial}{\partial x_4},$$
$$X_4 = [X_1, X_3] = \frac{\partial}{\partial x_4}.$$

Thus, we have

$$X_1(x) = \left(1,0, -\frac{x_2}{2}, -\left(\frac{x_3}{2} - \frac{x_1 x_2}{12}\right)\right),$$
$$X_2(x) = \left(0,1, \frac{x_1}{2}, \frac{x_1^2}{12}\right).$$

A direct calculation gives that

$$\langle X_1(x), \nu\rangle = \nu_1 - \frac{x_2}{2}\nu_3 - \left(\frac{x_3}{2} - \frac{x_1 x_2}{12}\right)\nu_4,$$
$$\langle X_2(x), \nu\rangle = \nu_2 + \frac{x_1}{2}\nu_3 + \frac{x_1^2}{12}\nu_4,$$
$$X_1\langle X_1(x), \nu\rangle = \frac{x_2}{12}\nu_4 + \frac{x_2}{4}\nu_4 = \frac{x_2 \nu_4}{3},$$
$$X_2\langle X_2(x), \nu\rangle = 0.$$

Now substituting these into inequality (2.5) we obtain the desired result.

2.2 Geometric L^p-Hardy type inequality on half-spaces

Here we construct an L^p version of the geometric Hardy inequality on the half-space of $\mathbb{G}$ as a generalisation of the previous theorem. We define the p-version of the angle function by $\mathcal{W}_p$, which is given by the formula

$$\mathcal{W}_p(x) = \left(\sum_{i=1}^N |\langle X_i(x), \nu\rangle|^p\right)^{\frac{1}{p}}. \tag{2.15}$$

Theorem 2.2.1 *Let $\mathbb{G}^+$ be a half-space of a stratified group $\mathbb{G}$. Then for all $\beta \in \mathbb{R}$ we have*

$$\int_{\mathbb{G}^+} \sum_{i=1}^{N} |X_i u|^p dx \geq C_2(\beta,p) \int_{\mathbb{G}^+} \frac{\mathcal{W}_p(x)^p}{dist(x,\partial\mathbb{G}^+)^p} |u|^p dx$$
$$+\beta(p-1) \int_{\mathbb{G}^+} \sum_{i=1}^{N} \left(\frac{|\langle X_i(x),\nu\rangle|}{dist(x,\partial\mathbb{G}^+)}\right)^{p-2} \frac{X_i\langle X_i(x),\nu\rangle}{dist(x,\partial\mathbb{G}^+)} |u|^p dx \qquad (2.16)$$

for all $u \in C_0^\infty(\mathbb{G}^+)$, $1 < p < \infty$ *and* $C_2(\beta,p) := -(p-1)(|\beta|^{\frac{p}{p-1}} + \beta)$.

Proof of Theorem 2.2.1. We use the standard method such as the divergence theorem to obtain the inequality (2.16). For $W \in C^\infty(\mathbb{G}^+)$ and $f \in C^1(\mathbb{G}^+)$, a direct calculation shows that

$$\int_{\mathbb{G}^+} \mathrm{d}iv_{\mathbb{G}}(fW)|u|^p dx = -\int_{\mathbb{G}^+} fW \cdot \nabla_{\mathbb{G}} |u|^p dx$$
$$= -p \int_{\mathbb{G}^+} f\langle W, \nabla_{\mathbb{G}} u\rangle |u|^{p-1} dx$$
$$\leq p\left(\int_{\mathbb{G}^+} |\langle W, \nabla_{\mathbb{G}} u\rangle|^p dx\right)^{\frac{1}{p}} \left(\int_{\mathbb{G}^+} |f|^{\frac{p}{p-1}} |u|^p dx\right)^{\frac{p-1}{p}}. \qquad (2.17)$$

Here in the last line Hölder's inequality was applied. For $p > 1$ and $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ recall Young's inequality

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}, \text{for } a \geq 0, b \geq 0.$$

Let us set that

$$a := \left(\int_{\mathbb{G}^+} |\langle W, \nabla_{\mathbb{G}} u\rangle|^p dx\right)^{\frac{1}{p}} \quad \text{and} \quad b := \left(\int_{\mathbb{G}^+} |f|^{\frac{p}{p-1}} |u|^p dx\right)^{\frac{p-1}{p}}.$$

By using Young's inequality in (2.17) and rearranging the terms, we arrive at

$$\int_{\mathbb{G}^+} |\langle W, \nabla_{\mathbb{G}} u\rangle|^p dx \geq \int_{\mathbb{G}^+} \left(\mathrm{d}iv_{\mathbb{G}}(fW) - (p-1)|f|^{\frac{p}{p-1}}\right) |u|^p dx. \qquad (2.18)$$

We choose $W := I_i$, which has the following form $I_i = (\overbrace{0, \ldots, 1}^{i} \ldots, 0)$ and set

$$f = \beta \frac{|\langle X_i(x), \nu\rangle|^{p-1}}{dist(x,\partial\mathbb{G}^+)^{p-1}}.$$

Now we calculate

$$\mathrm{d}iv_{\mathbb{G}}(Wf) = (\nabla_{\mathbb{G}} \cdot I_i) f = X_i f = \beta X_i \left(\frac{|\langle X_i(x),\nu\rangle|}{dist(x,\partial\mathbb{G}^+)}\right)^{p-1}$$

$$= \beta(p-1)\left(\frac{|\langle X_i(x),\nu\rangle|}{dist(x,\partial\mathbb{G}^+)}\right)^{p-2} X_i\left(\frac{\langle X_i(x),\nu\rangle}{dist(x,\partial\mathbb{G}^+)}\right)$$
$$= \beta(p-1)\left(\frac{|\langle X_i(x),\nu\rangle|}{dist(x,\partial\mathbb{G}^+)}\right)^{p-2}\left(\frac{X_i\langle X_i(x),\nu\rangle}{dist(x,\partial\mathbb{G}^+)} - \frac{|\langle X_i(x),\nu\rangle|^2}{dist(x,\partial\mathbb{G}^+)^2}\right)$$
$$= \beta(p-1)\left[\left(\frac{|\langle X_i(x),\nu\rangle|}{dist(x,\partial\mathbb{G}^+)}\right)^{p-2}\left(\frac{X_i\langle X_i(x),\nu\rangle}{dist(x,\partial\mathbb{G}^+)}\right) - \frac{|\langle X_i(x),\nu\rangle|^p}{dist(x,\partial\mathbb{G}^+)^p}\right],$$

and

$$|f|^{\frac{p}{p-1}} = |\beta|^{\frac{p}{p-1}}\frac{|\langle X_i(x),\nu\rangle|^p}{dist(x,\partial\mathbb{G}^+)^p}.$$

We also have

$$\langle W,\nabla_{\mathbb{G}}u\rangle = (\overbrace{0,\dots,1}^{i}\dots,0)\cdot(X_1u,\dots,X_iu,\dots,X_Nu)^T = X_iu.$$

Inserting the above calculations in (2.18) and summing over $i=1,\dots,N$, we arrive at

$$\int_{\mathbb{G}^+}\sum_{i=1}^{N}|X_iu|^p dx \geq -(p-1)(|\beta|^{\frac{p}{p-1}}+\beta)\int_{\mathbb{G}^+}\sum_{i=1}^{N}\frac{|\langle X_i(x),\nu\rangle|^p}{dist(x,\partial\mathbb{G}^+)^p}|u|^p dx$$
$$+\beta(p-1)\int_{\mathbb{G}^+}\sum_{i=1}^{N}\left(\frac{|\langle X_i(x),\nu\rangle|}{dist(x,\partial\mathbb{G}^+)}\right)^{p-2}\frac{X_i\langle X_i(x),\nu\rangle}{dist(x,\partial\mathbb{G}^+)}|u|^p dx. \qquad (2.19)$$

We complete the proof of Theorem 2.2.1.

Remark 2.2.2 For $p\geq 2$, since

$$|\nabla_{\mathbb{G}}u|^p = \left(\sum_{i=1}^{N}|X_iu|^2\right)^{\frac{p}{2}} \geq \sum_{i=1}^{N}(|X_iu|^2)^{\frac{p}{2}}, \qquad (2.20)$$

we have the following inequality

$$\int_{\mathbb{G}^+}|\nabla_{\mathbb{G}}u|^p dx \geq C_2(\beta,p)\int_{\mathbb{G}^+}\frac{\mathcal{W}_p(x)^p}{dist(x,\partial\mathbb{G}^+)^p}|u|^p dx \qquad (2.21)$$
$$+\beta(p-1)\int_{\mathbb{G}^+}\sum_{i=1}^{N}\left(\frac{|\langle X_i(x),\nu\rangle|}{dist(x,\partial\mathbb{G}^+)}\right)^{p-2}\frac{X_i\langle X_i(x),\nu\rangle}{dist(x,\partial\mathbb{G}^+)}|u|^p dx.$$

2.3 Geometric L^2-Hardy inequality on convex domains

In this section, we present the geometric Hardy inequalities on the convex domains in stratified groups. The convex domain is understood in the sense of the Euclidean space. Let Ω be a convex domain of a stratified group $\mathbb{G}$ and let $\partial\Omega$ be its boundary. Below for $x\in\Omega$ we denote by $\nu(x)$ the unit normal for $\partial\Omega$ at a point $\hat{x}\in\partial\Omega$ such that $dist(x,\Omega)=dist(x,\hat{x})$. For the half-plane, we have the distance from the boundary $dist(x,\partial\Omega)=\langle x,\nu\rangle-d$. As it is introduced in the previous section we also have the generalised angle function

$$\mathcal{W}_p(x) = \left(\sum_{i=1}^{N} |\langle X_i(x), \nu \rangle|^p \right)^{\frac{1}{p}},$$

with $\mathcal{W}(x) := \mathcal{W}_2(x)$.

Theorem 2.3.1 *Let Ω be a convex domain of a stratified group $\mathbb{G}$. Then for $\beta < 0$ we have*

$$\int_{\Omega} |\nabla_{\mathbb{G}} u|^2 dx \geq C_1(\beta) \int_{\Omega} \frac{\mathcal{W}(x)^2}{dist(x, \partial\Omega)^2} |u|^2 dx + \beta \int_{\Omega} \sum_{i=1}^{N} \frac{X_i \langle X_i(x), \nu \rangle}{dist(x, \partial\Omega)} |u|^2 dx, \quad (2.22)$$

for all $u \in C_0^\infty(\Omega)$, and $C_1(\beta) := -(\beta^2 + \beta)$.

Proof of Theorem 2.3.1. We follow the approach of Larson by proving inequality (2.22) in the case when Ω is a convex polytope. We denote its facets by $\{\mathcal{F}_j\}_j$ and unit normals of these facets by $\{\nu_j\}_j$, which are directed inward. Then Ω can be constructed by the union of the disjoint sets $\Omega_j := \{x \in \Omega: dist(x, \partial\Omega) = dist(x, \mathcal{F}_j)\}$. Now we apply the same method as in the case of the half-space $\mathbb{G}^+$ for each element Ω_j with one exception that not all the boundary values are zero when we use the partial integration. As in the previous computation we have

$$\begin{aligned}
0 &\leq \int_{\Omega_j} |\nabla_{\mathbb{G}} u + \beta W u|^2 dx = \int_{\Omega_j} \sum_{i=1}^{N} |X_i u + \beta W_i u|^2 dx \\
&= \int_{\Omega_j} \sum_{i=1}^{N} [|X_i u|^2 + 2\mathrm{Re} \beta W_i u X_i u + \beta^2 W_i^2 |u|^2] dx \\
&= \int_{\Omega_j} \sum_{i=1}^{N} [|X_i u|^2 + \beta W_i X_i |u|^2 + \beta^2 W_i^2 |u|^2] dx \\
&= \int_{\Omega_j} \sum_{i=1}^{N} [|X_i u|^2 - \beta (X_i W_i) |u|^2 + \beta^2 W_i^2 |u|^2] dx \\
&+ \beta \int_{\partial\Omega_j} \sum_{i=1}^{N} W_i \langle X_i(x), n_j(x) \rangle |u|^2 d\Gamma_{\partial\Omega_j}(x),
\end{aligned}$$

where n_j is the unit normal of $\partial\Omega_j$ which is directed outward. Since $\mathcal{F}_j \subset \partial\Omega_j$ we have $n_j = -\nu_j$.

The boundary terms on $\partial\Omega$ vanish since u is compactly supported in Ω. So we only deal with the parts of $\partial\Omega_j$ in Ω. Note that for every facet of $\partial\Omega_j$ there exists some $\partial\Omega_l$ which shares this facet. We denote by Γ_{jl} the common facet of $\partial\Omega_j$ and $\partial\Omega_l$, with $n_k|_{\Gamma_{jl}} = -n_l|_{\Gamma_{jl}}$. From the above expression we get the following inequality

$$\begin{aligned}
\int_{\Omega_j} |\nabla_{\mathbb{G}} u|^2 dx &\geq \int_{\Omega_j} \sum_{i=1}^{N} [(\beta (X_i W_i) - \beta^2 W_i^2) |u|^2] dx \quad (2.23) \\
&- \beta \int_{\partial\Omega_j} \sum_{i=1}^{N} W_i \langle X_i(x), n_j(x) \rangle |u|^2 d\Gamma_{\partial\Omega_j}(x).
\end{aligned}$$

Now we choose W_i in the form

$$W_i(x) = \frac{\langle X_i(x), \nu_j\rangle}{dist(x, \partial\Omega_j)} = \frac{\langle X_i(x), \nu_j\rangle}{\langle x, \nu_j\rangle - d},$$

and a direct computation shows that

$$X_i W_i(x) = \frac{X_i\langle X_i(x),\nu_j\rangle}{dist(x,\partial\Omega_j)} - \frac{\langle X_i(x),\nu_j\rangle^2}{dist(x,\partial\Omega_j)^2}. \tag{2.24}$$

Inserting the expression (2.24) into inequality (2.23) we get

$$\int_{\Omega_j} |\nabla_{\mathbb{G}} u|^2 dx \geq -(\beta^2+\beta)\int_{\Omega_j} \sum_{i=1}^N \frac{\langle X_i(x),\nu_j\rangle^2}{dist(x,\partial\Omega_j)^2} |u|^2 dx \tag{2.25}$$

$$+\beta \int_{\Omega_j} \sum_{i=1}^N \frac{X_i\langle X_i(x),\nu_j\rangle}{dist(x,\partial\Omega_j)} |u|^2 dx - \beta \int_{\Gamma_{jl}} \sum_{i=1}^N \frac{\langle X_i(x),\nu_j\rangle\langle X_i(x),n_{jl}\rangle}{dist(x,\mathcal{F}_j)} |u|^2 d\Gamma_{jl}.$$

Now we sum over all partition elements Ω_j and let $n_{jl} = n_k|_{\Gamma_{jl}}$, i.e. the unit normal of Γ_{jl} pointing from Ω_j into Ω_l. Then we get

$$\begin{aligned}
&\int_{\Omega} |\nabla_{\mathbb{G}} u|^2 dx \geq -(\beta^2+\beta)\int_{\Omega} \sum_{i=1}^N \frac{\langle X_i(x),\nu\rangle^2}{dist(x,\partial\Omega)^2} |u|^2 dx \\
&+\beta \int_{\Omega} \sum_{i=1}^N \frac{X_i\langle X_i(x),\nu\rangle}{dist(x,\partial\Omega)} |u|^2 dx \\
&-\beta \sum_{j\neq l} \int_{\Gamma_{jl}} \sum_{i=1}^N \frac{\langle X_i(x),\nu_j\rangle\langle X_i(x),n_{jl}\rangle}{dist(x,\mathcal{F}_j)} |u|^2 d\Gamma_{jl} \\
&= -(\beta^2+\beta)\int_{\Omega} \sum_{i=1}^N \frac{\langle X_i(x),\nu\rangle^2}{dist(x,\partial\Omega)^2} |u|^2 dx \\
&+\beta \int_{\Omega} \sum_{i=1}^N \frac{X_i\langle X_i(x),\nu\rangle}{dist(x,\partial\Omega)} |u|^2 dx \\
&-\beta \sum_{j<l} \int_{\Gamma_{jl}} \sum_{i=1}^N \frac{\langle X_i(x),\nu_j-\nu_l\rangle\langle X_i(x),n_{jl}\rangle}{dist(x,\mathcal{F}_j)} |u|^2 d\Gamma_{jl}.
\end{aligned}$$

Here we used the fact that (by the definition) Γ_{jl} is a set with $dist(x,\mathcal{F}_j) = dist(x,\mathcal{F}_l)$. From

$$\Gamma_{jl} = \{x : x\cdot\nu_j - d_j = x\cdot\nu_l - d_l\}$$

rearranging $x\cdot(\nu_j-\nu_l) - d_j + d_l = 0$ we see that Γ_{jl} is a hyperplane with a normal $\nu_j - \nu_l$. Thus, $\nu_j - \nu_l$ is parallel to n_{jl} and one only needs to check that $(\nu_j-\nu_l)\cdot n_{jl} > 0$. Observe that n_{jl} points out and ν_j points into j-th partition element, so $\nu_j\cdot n_{jl}$ is non-negative. Similarly, we see that $\nu_l\cdot n_{jl}$ is non-positive. This means we have $(\nu_j-\nu_l)\cdot n_{jl} > 0$. In addition, it is easy to see that

$$|\nu_j - \nu_l|^2 = (\nu_j - \nu_l)\cdot(\nu_j - \nu_l) = 2 - 2\nu_j \cdot \nu_l$$
$$= 2 - 2\cos(\alpha_{jl}),$$

which implies that

$$(\nu_j - \nu_l)\cdot n_{jl} = \sqrt{2 - 2\cos(\alpha_{jl})},$$

where α_{jl} is the angle between ν_j and ν_l. So we obtain

$$\int_\Omega |\nabla_{\mathbb{G}} u|^2 dx \geq -(\beta^2+\beta)\int_\Omega \sum_{i=1}^N \frac{\langle X_i(x),\nu\rangle^2}{dist(x,\partial\Omega)^2}|u|^2 dx$$
$$+\beta\int_\Omega \sum_{i=1}^N \frac{X_i\langle X_i(x),\nu\rangle}{dist(x,\partial\Omega)}|u|^2 dx$$
$$-\beta\sum_{j<l}\sum_{i=1}^N \int_{\Gamma_{jl}} \sqrt{1-\cos(\alpha_{jl})}\frac{\langle X_i(x),n_{jl}\rangle^2}{dist(x,\mathcal{F}_j)}|u|^2 d\Gamma_{jl}.$$

Here with $\beta < 0$ and due to the boundary term signs we verify the inequality for the polytope convex domains.

Let us now consider the general case, that is, when Ω is an arbitrary convex domain. For each $u \in C_0^\infty(\Omega)$ one can always choose an increasing sequence of convex polytopes $\{\Omega_j\}_{j=1}^\infty$ such that $u \in C_0^\infty(\Omega_1), \Omega_j \subset \Omega$ and $\Omega_j \to \Omega$ as $j \to \infty$. Assume that $\nu_j(x)$ is the above map ν (corresponding to Ω_j) we compute

$$\int_\Omega |\nabla_{\mathbb{G}} u|^2 dx = \int_{\Omega_j} |\nabla_{\mathbb{G}} u|^2 dx$$
$$\geq -(\beta^2+\beta)\int_{\Omega_j}\sum_{i=1}^N \frac{\langle X_i(x),\nu_j\rangle^2}{dist(x,\partial\Omega_j)^2}|u|^2 dx + \beta\int_{\Omega_j}\sum_{i=1}^N \frac{X_i\langle X_i(x),\nu_j\rangle}{dist(x,\partial\Omega_j)}|u|^2 dx$$
$$= -(\beta^2+\beta)\int_{\Omega}\sum_{i=1}^N \frac{\langle X_i(x),\nu_j\rangle^2}{dist(x,\partial\Omega_j)^2}|u|^2 dx + \beta\int_{\Omega}\sum_{i=1}^N \frac{X_i\langle X_i(x),\nu_j\rangle}{dist(x,\partial\Omega_j)}|u|^2 dx$$
$$\geq -(\beta^2+\beta)\int_{\Omega}\sum_{i=1}^N \frac{\langle X_i(x),\nu_j\rangle^2}{dist(x,\partial\Omega)^2}|u|^2 dx + \beta\int_{\Omega}\sum_{i=1}^N \frac{X_i\langle X_i(x),\nu_j\rangle}{dist(x,\partial\Omega)}|u|^2 dx.$$

Now we obtain the desired result when $j \to \infty$.

2.4 Geometric L^p-Hardy's inequality on convex domains. In this section we give the L^p-version of the previous results.

Theorem 2.4.1 *Let Ω be a convex domain of a stratified group $\mathbb{G}$. Then for $\beta < 0$ we have*

$$\int_\Omega \sum_{i=1}^N |X_i u|^p dx \geq C_2(\beta,p)\int_\Omega \frac{\mathcal{W}_p(x)^p}{dist(x,\partial\Omega)^p}|u|^p dx \tag{2.26}$$
$$+\beta(p-1)\int_\Omega \sum_{i=1}^N \left(\frac{|\langle X_i(x),\nu\rangle|}{dist(x,\partial\Omega)}\right)^{p-2}\left(\frac{X_i\langle X_i(x),\nu\rangle}{dist(x,\partial\Omega)}\right)|u|^p dx,$$

for all $u \in C_0^\infty(\Omega)$, *and* $C_2(\beta,p) := -(p-1)(|\beta|^{\frac{p}{p-1}} + \beta)$.

P r o o f o f T h e o r e m 2 . 4 . 1 . Let us assume that Ω is the convex polytope as in the $p=2$ case. Thus, we consider the partition Ω_j as the previous case. For $f \in C^1(\Omega_j)$ and $W \in C^\infty(\Omega_j)$, a simple calculation shows that

$$\int_{\Omega_j} \mathrm{d}iv_{\mathbb{G}}(fW)|u|^p dx$$
$$= -p\int_{\Omega_j} f\langle W, \nabla_{\mathbb{G}} u\rangle |u|^{p-1} dx + \int_{\partial\Omega_j} f\langle W, n_j(x)\rangle |u|^p d\Gamma_{\partial\Omega_j}(x)$$
$$\leq p\big(\textstyle\int_\Omega |\langle W, \nabla_{\mathbb{G}} u\rangle|^p dx\big)^{\frac{1}{p}} \left(\textstyle\int_{\Omega_j} |f|^{\frac{p}{p-1}} |u|^p dx\right)^{\frac{p-1}{p}} \tag{2.27}$$
$$+\int_{\partial\Omega_j} f\langle W, n_j(x)\rangle |u|^p d\Gamma_{\partial\Omega_j}(x).$$

In the last line Hölder's inequality was applied. Recall again Young's inequality for $p > 1$, $q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, we have $ab \leq \frac{a^p}{p} + \frac{b^q}{q}$, for $a \geq 0, b \geq 0$. We now take $q := \frac{p}{p-1}$ and

$$a := \big(\textstyle\int_\Omega |\langle W, \nabla_{\mathbb{G}} u\rangle|^p dx\big)^{\frac{1}{p}} \quad \text{and} \quad b := \left(\textstyle\int_\Omega |f|^{\frac{p}{p-1}} |u|^p dx\right)^{\frac{p-1}{p}}.$$

By using Young's inequality in (2.27) and rearranging the terms, we arrive at

$$\textstyle\int_{\Omega_j} |\langle W, \nabla_{\mathbb{G}} u\rangle|^p dx \geq \int_\Omega \left(\mathrm{d}iv_{\mathbb{G}}(fW) - (p-1)|f|^{\frac{p}{p-1}}\right)|u|^p dx \tag{2.28}$$
$$-\textstyle\int_{\partial\Omega_j} f\langle W, n_j(x)\rangle |u|^p d\Gamma_{\partial\Omega_j}(x).$$

We choose $W := I_i$ as a unit vector of the i^{th} component and let

$$f = \beta \frac{|\langle X_i(x), \nu_j\rangle|^{p-1}}{dist(x, \mathcal{F}_j)^{p-1}}.$$

As before a direct calculation shows that

$$\mathrm{d}iv_{\mathbb{G}}(Wf) = X_i f = \beta X_i \left(\frac{|\langle X_i(x), \nu_j\rangle|}{dist(x, \partial\mathcal{F}_j)}\right)^{p-1}$$
$$= \beta(p-1)\left(\frac{|\langle X_i(x), \nu_j\rangle|}{dist(x, \partial\mathcal{F}_j)}\right)^{p-2} X_i \left(\frac{\langle X_i(x), \nu_j\rangle}{dist(x, \partial\mathcal{F}_j)}\right)$$

$$= \beta(p-1)\left(\frac{|\langle X_i(x),\nu_j\rangle|}{dist(x,\partial\mathcal{F}_j)}\right)^{p-2}\left(\frac{X_i\langle X_i(x),\nu_j\rangle}{dist(x,\partial\mathcal{F}_j)}-\frac{|\langle X_i(x),\nu_j\rangle|^2}{dist(x,\partial\mathcal{F}_j)^2}\right)$$
$$= \beta(p-1)\left[\left(\frac{|\langle X_i(x),\nu_j\rangle|}{dist(x,\partial\mathcal{F}_j)}\right)^{p-2}\left(\frac{X_i\langle X_i(x),\nu_j\rangle}{dist(x,\partial\mathcal{F}_j)}\right)-\frac{|\langle X_i(x),\nu_j\rangle|^p}{dist(x,\partial\mathcal{F}_j)^p}\right],$$

and

$$|f|^{\frac{p}{p-1}} = |\beta|^{\frac{p}{p-1}}\frac{|\langle X_i(x),\nu_j\rangle|^p}{dist(x,\mathcal{F}_j)^p}.$$

We also have

$$\langle W,\nabla_{\mathbb{G}}u\rangle = (\overbrace{0,\dots,1}^{i}\dots,0)\cdot(X_1u,\dots,X_iu,\dots,X_Nu)^T = X_iu.$$

Inserting the above calculations into (2.28) and summing over $i=\overline{1,N}$, we arrive at

$$\int_{\Omega_j}\sum_{i=1}^N|X_iu|^p dx \geq -(p-1)(|\beta|^{\frac{p}{p-1}}+\beta)\int_{\Omega_j}\sum_{i=1}^N\frac{|\langle X_i(x),\nu_j\rangle|^p}{dist(x,\partial\mathcal{F}_j)^p}|u|^p dx \quad (2.29)$$
$$+\beta(p-1)\int_{\Omega_j}\sum_{i=1}^N\left(\frac{|\langle X_i(x),\nu_j\rangle|}{dist(x,\partial\mathcal{F}_j)}\right)^{p-2}\left(\frac{X_i\langle X_i(x),\nu_j\rangle}{dist(x,\partial\mathcal{F}_j)}\right)|u|^p dx$$
$$-\beta\int_{\partial\Omega_j}\sum_{i=1}^N\left(\frac{|\langle X_i(x),\nu_j\rangle|}{dist(x,\mathcal{F}_j)}\right)^{p-1}\langle X_i(x),n_j(x)\rangle|u|^p d\Gamma_{\partial\Omega_j}(x).$$

Now summing up over Ω_j, and with the interior boundary terms we have

$$\int_\Omega\sum_{i=1}^N|X_iu|^p dx \geq -(p-1)(|\beta|^{\frac{p}{p-1}}+\beta)\sum_{i=1}^N\int_\Omega\frac{|\langle X_i(x),\nu\rangle|^p}{dist(x,\partial\Omega)^p}|u|^p dx$$
$$+\beta(p-1)\sum_{i=1}^N\int_\Omega\left(\frac{|\langle X_i(x),\nu\rangle|}{dist(x,\partial\Omega)}\right)^{p-2}\left(\frac{X_i\langle X_i(x),\nu\rangle}{dist(x,\partial\Omega)}\right)|u|^p dx$$
$$-\beta\sum_{j\neq l}\sum_{i=1}^N\int_{\Gamma_{jl}}\left(\frac{|\langle X_i(x),\nu_j\rangle|}{dist(x,\mathcal{F}_j)}\right)^{p-1}\langle X_i(x),n_{jl}(x)\rangle|u|^p d\Gamma_{jl}$$
$$= -(p-1)(|\beta|^{\frac{p}{p-1}}+\beta)\sum_{i=1}^N\int_\Omega\frac{|\langle X_i(x),\nu\rangle|^p}{dist(x,\partial\Omega)^p}|u|^p dx$$
$$+\beta(p-1)\sum_{i=1}^N\int_\Omega\left(\frac{|\langle X_i(x),\nu\rangle|}{dist(x,\partial\Omega)}\right)^{p-2}\left(\frac{X_i\langle X_i(x),\nu\rangle}{dist(x,\partial\Omega)}\right)|u|^p dx$$
$$-\beta\sum_{j<l}\sum_{i=1}^N\int_{\Gamma_{jl}}\left[\left(\frac{|\langle X_i(x),\nu_j\rangle|}{dist(x,\mathcal{F}_j)}\right)^{p-1}\langle X_i(x),n_{jl}(x)\rangle\right.$$
$$\left.-\left(\frac{|\langle X_i(x),\nu_l\rangle|}{dist(x,\mathcal{F}_l)}\right)^{p-1}\langle X_i(x),n_{jl}(x)\rangle\right]|u|^p d\Gamma_{jl}.$$

As in the earlier case if the boundary term is positive we can discard it, so we want to show that

$$\left[\left(\frac{|\langle X_i(x),\nu_j\rangle|}{dist(x,\mathcal{F}_j)}\right)^{p-1}\langle X_i(x),n_{jl}(x)\rangle-\left(\frac{|\langle X_i(x),\nu_l\rangle|}{dist(x,\mathcal{F}_l)}\right)^{p-1}\langle X_i(x),n_{jl}(x)\rangle\right]\geq 0.$$

Noting the fact that $n_{jl}=\frac{\nu_j-\nu_l}{\sqrt{2-2\cos(\alpha_{jl})}}$ and $dist(x,\mathcal{F}_j)=dist(x,\mathcal{F}_l)$ on Γ_{jl}, we arrive at

$$\frac{1}{2-2\cos(\alpha_{jl})}\left[\left(\frac{|\langle X_i(x),\nu_j\rangle|}{dist(x,\mathcal{F}_j)}\right)^{p-1}\langle X_i(x),\nu_j-\nu_l\rangle-\left(\frac{|\langle X_i(x),\nu_l\rangle|}{dist(x,\mathcal{F}_l)}\right)^{p-1}\langle X_i(x),\nu_j-\nu_l\rangle\right]$$
$$=\frac{|\langle X_i(x),\nu_j\rangle|^p-|\langle X_i(x),\nu_j\rangle|^{p-1}\langle X_i(x),\nu_l\rangle-|\langle X_i(x),\nu_l\rangle|^{p-1}\langle X_i(x),\nu_j\rangle+|\langle X_i(x),\nu_l\rangle|^p}{(2-2\cos(\alpha_{jl}))dist(x,\mathcal{F}_j)^{p-1}}$$
$$=\frac{\left(|\langle X_i(x),\nu_j\rangle|-|\langle X_i(x),\nu_l\rangle|\right)\left(|\langle X_i(x),\nu_j\rangle|^{p-1}-|\langle X_i(x),\nu_l\rangle|^{p-1}\right)}{(2-2\cos(\alpha_{jl}))dist(x,\mathcal{F}_j)^{p-1}}\geq 0.$$

Here we have used the equality $(a-b)(a^{p-1}-b^{p-1})=a^p-a^{p-1}b-b^{p-1}a+b^{p-1}$ with $a=|\langle X_i(x),\nu_j\rangle|$ and $b=|\langle X_i(x),\nu_l\rangle|$. From the above expression we note that the boundary term in Ω is positive and $\beta<0$. By discarding the boundary term we complete the proof.

Remark 2.4.2 For $p\geq 2$, since

$$|\nabla_{\mathbb{G}}u|^p=\left(\sum_{i=1}^N|X_iu|^2\right)^{\frac{p}{2}}\geq\sum_{i=1}^N(|X_iu|^2)^{\frac{p}{2}},\tag{2.30}$$

we have the following inequality

$$\int_\Omega|\nabla_{\mathbb{G}}u|^p dx\geq C_2(\beta,p)\int_\Omega\frac{\mathcal{W}_p(x)^p}{dist(x,\partial\Omega)^p}|u|^p dx\tag{2.31}$$
$$+\beta(p-1)\int_\Omega\sum_{i=1}^N\left(\frac{|\langle X_i(x),\nu\rangle|}{dist(x,\partial\Omega)}\right)^{p-2}\left(\frac{X_i\langle X_i(x),\nu\rangle}{dist(x,\partial\Omega)}\right)|u|^p dx.$$

2.5 Geometric L^p-Hardy inequality with a natural weight

Theorem 2.5.1 *Let $\mathbb{G}^+$ be a half-space of a stratified group $\mathbb{G}$. Then for all $\beta\in\mathbb{R}$ and $p>1$ we have*

$$\int_{\mathbb{G}^+}|\nabla_{\mathbb{G}}u|^p dx\geq-(p-1)(|\beta|^{\frac{p}{p-1}}+\beta)\int_{\mathbb{G}^+}\frac{\mathcal{W}(x)^p}{dist(x,\partial\mathbb{G}^+)^p}|u|^p dx\tag{2.32}$$
$$+\beta\int_{\mathbb{G}^+}\frac{\mathcal{L}_p(dist(x,\partial\mathbb{G}^+))}{dist(x,\partial\mathbb{G}^+)^{p-1}}|u|^p dx,$$

for all $u\in C_0^\infty(\mathbb{G}^+)$.

P r o o f o f T h e o r e m 2 . 5 . 1 . Let us begin with the divergence theorem, then we apply the Hölder inequality and the Young inequality, respectively. It follows for a vector field $V \in C^{\infty}(\mathbb{G}^{+})$ that

$$\int_{\mathbb{G}^+} \mathrm{d}iv_{\mathbb{G}} V |u|^p dx = -p \int_{\mathbb{G}^+} |u|^{p-1} \langle V, \nabla_{\mathbb{G}} u \rangle dx$$
$$\leq p \left(\int_{\mathbb{G}^+} |\nabla_{\mathbb{G}} u|^p dx \right)^{\frac{1}{p}} \left(\int_{\mathbb{G}^+} |V|^{\frac{p}{p-1}} |u|^p dx \right)^{\frac{p-1}{p}}$$
$$\leq \int_{\mathbb{G}^+} |\nabla_{\mathbb{G}} u|^p dx + (p-1) \int_{\mathbb{G}^+} |V|^{\frac{p}{p-1}} |u|^p dx.$$

By rearranging the above expression, we arrive at

$$\int_{\mathbb{G}^+} |\nabla_{\mathbb{G}} u|^p dx \geq \int_{\mathbb{G}^+} \left(\mathrm{d}iv_{\mathbb{G}} V - (p-1) |V|^{\frac{p}{p-1}} \right) |u|^p dx. \tag{2.33}$$

Now we choose V in the following form

$$V = \beta \frac{|\nabla_{\mathbb{G}} dist(x, \partial\mathbb{G}^+)|^{p-2}}{dist(x, \partial\mathbb{G}^+)^{p-1}} \nabla_{\mathbb{G}} dist(x, \partial\mathbb{G}^+), \tag{2.34}$$

that is

$$|V|^{\frac{p}{p-1}} = |\beta|^{\frac{p}{p-1}} \frac{|\nabla_{\mathbb{G}} dist(x, \partial\mathbb{G}^+)|^p}{dist(x, \partial\mathbb{G}^+)^p}.$$

Also, we have

$$|\nabla_{\mathbb{G}} dist(x, \partial\mathbb{G}^+)|^p = |(X_1 dist(x, \partial\mathbb{G}^+), \ldots, X_N dist(x, \partial\mathbb{G}^+))|^p$$
$$= |(\langle X_1(x), \nu \rangle, \ldots, \langle X_N(x), \nu \rangle)|^p$$
$$= \left(\sum_{i=1}^{N} \langle X_i(x), \nu \rangle^2 \right)^{\frac{p}{2}} = \mathcal{W}(x)^p.$$

Indeed, let us show that $\langle X_i(x), \nu \rangle = X_i \langle x, \nu \rangle$:

$$X_i(x) = ((\overbrace{0, \ldots, 1}^{i} \ldots, 0, \underbrace{a_{i,1}^{(2)}(x'), \ldots, a_{i,N_2}^{(2)}(x')}_{N_2}, \ldots,$$
$$\underbrace{a_{i,1}^{(r)}(x', x^{(2)}, \ldots, x^{(r-1)}), \ldots a_{i,N_r}^{(r)}(x', x^{(2)}, \ldots, x^{(r-1)})}_{N_r}),$$
$$\langle X_i(x), \nu \rangle = \nu'_i + \sum_{l=2}^{r} \sum_{m=1}^{N_l} a_{i,m}^{(l)}(x', x^{(2)}, \ldots, x^{(r-1)}) \nu_m^{(l)},$$

and

$$\langle x, \nu \rangle = \sum_{k=1}^{N} x_{k'} \nu'_k + \sum_{l=2}^{r} \sum_{m=1}^{N_l} x_m^{(l)} \nu_m^{(l)},$$

$$X_i\langle x,\nu\rangle = \nu'_i + \sum_{l=2}^{r}\sum_{m=1}^{N_l} a_{i,m}^{(l)}(x', x^{(2)}, \dots, x^{(r-1)})\nu_m^{(l)}.$$

A direct calculation shows that

$$\mathrm{d}iv_{\mathbb{G}}V = \beta \frac{\nabla_{\mathbb{G}}(|\nabla_{\mathbb{G}} dist(x,\partial\mathbb{G}^+)|^{p-2}\nabla_{\mathbb{G}} dist(x,\partial\mathbb{G}^+))}{dist(x,\partial\mathbb{G}^+)^{p-1}}$$
$$-\beta(p-1)\frac{|\nabla_{\mathbb{G}} dist(x,\partial\mathbb{G}^+)|^{p-2}\nabla_{\mathbb{G}} dist(x,\partial\mathbb{G}^+) dist(x,\partial\mathbb{G}^+)^{p-2}\nabla_{\mathbb{G}} dist(x,\partial\mathbb{G}^+)}{dist(x,\partial\mathbb{G}^+)^{2(p-1)}}$$
$$= \beta\frac{\mathcal{L}_p(dist(x,\partial\mathbb{G}^+))}{dist(x,\partial\mathbb{G}^+)^{p-1}} - \beta(p-1)\frac{|\nabla_{\mathbb{G}} dist(x,\partial\mathbb{G}^+)|^p}{dist(x,\partial\mathbb{G}^+)^p}.$$

So we get

$$\mathrm{d}iv_{\mathbb{G}}V - (p-1)|V|^{\frac{p}{p-1}} = -(p-1)(|\beta|^{\frac{p}{p-1}} + \beta)\frac{|\nabla_{\mathbb{G}} dist(x,\partial\mathbb{G}^+)|^p}{dist(x,\partial\mathbb{G}^+)^p}$$
$$+\beta\frac{\mathcal{L}_p(dist(x,\partial\mathbb{G}^+))}{dist(x,\partial\mathbb{G}^+)^{p-1}}.$$

Putting the above expression into inequality (2.33), we arrive at

$$\int_{\mathbb{G}^+}|\nabla_{\mathbb{G}}u|^p dx \geq -(p-1)(|\beta|^{\frac{p}{p-1}}+\beta)\int_{\mathbb{G}^+}\frac{(\sum_{i=1}^N \langle X_i(x),\nu\rangle^2)^{\frac{p}{2}}}{dist(x,\partial\mathbb{G}^+)^p}|u|^p dx$$
$$+\beta\int_{\mathbb{G}^+}\frac{\mathcal{L}_p(dist(x,\partial\mathbb{G}^+))}{dist(x,\partial\mathbb{G}^+)^{p-1}}|u|^p dx,$$

completing the proof.

As a consequence of Theorem 2.5.1 we have the following inequality.

Corollary 2.5.2 *Let $\mathbb{H}^+$ be a half-space of the Heisenberg group $\mathbb{H}^n$. Then for all $u \in C_0^\infty(\mathbb{H}^+)$ and $p > 1$ we have*

$$\int_{\mathbb{H}^+}|\nabla_H u|^p d\xi \geq \left(\frac{p-1}{p}\right)^p \int_{\mathbb{H}^+}\frac{\mathcal{W}(\xi)^p}{dist(\xi,\partial\mathbb{H}^+)^p}|u|^p d\xi, \quad (2.35)$$

where the constant is sharp.

Remark 2.5.3 Note that inequality (2.35) was conjectured in [20, P. 337-338] which is a natural extension of inequality (2.2) in [40, P. 646-647]. Also, the sharpness of inequality (2.35) was proved by choosing $\nu := (1,0,\dots,0)$ and $d = 0$.

Proof of Corollary 2.5.2. Let us rewrite the inequality in Theorem 2.5.1 in terms of the Heisenberg group as follows

$$\int_{\mathbb{H}^+} |\nabla_H u|^p d\xi \geq -(p-1)(|\beta|^{\frac{p}{p-1}} + \beta) \int_{\mathbb{H}^+} \frac{\mathcal{W}(\xi)^p}{dist(\xi,\partial\mathbb{H}^+)^p} |u|^p d\xi$$
$$+\beta \int_{\mathbb{H}^+} \frac{\mathcal{L}_p(dist(\xi,\partial\mathbb{H}^+))}{dist(\xi,\partial\mathbb{H}^+)^{p-1}} |u|^p d\xi.$$

In the case of the Heisenberg group, we need to show that the last term vanishes to prove Corollary 2.5.1. Indeed, we have

$$\mathcal{L}_p(dist(\xi,\partial\mathbb{H}^+)) = 0,$$

since

$$\langle X_i(\xi), \nu\rangle = \nu_{x,i} + 2y_i\nu_t, \langle Y_i(\xi), \nu\rangle = \nu_{y,i} - 2x_i\nu_t,$$

$$X_i\langle X_i(\xi), \nu\rangle = 0, Y_i\langle Y_i(\xi), \nu\rangle = 0,$$

$$Y_i\langle X_i(\xi), \nu\rangle = 2\nu_t, X_i\langle Y_i(\xi), \nu\rangle = -2\nu_t,$$

where $\xi := (x, y, t)$ with $x, y \in \mathbb{R}^n$ and $t \in \mathbb{R}$, $\nu := (\nu_x, \nu_y, \nu_t)$ with $\nu_x := (\nu_{x,1}, \dots \nu_{x,n})$ and $\nu_y := (\nu_{y,1}, \dots \nu_{y,n})$. Then we have

$$X_i(\xi) = (\underbrace{\overbrace{0,\dots,1}^{i},\dots,0}_{n}, \underbrace{0,\dots,0}_{n}, 2y_i),$$
$$Y_i(\xi) = (\underbrace{0,\dots,0}_{n}, \underbrace{\overbrace{0,\dots,1}^{i},\dots,0}_{n}, -2x_i).$$

So we have

$$\int_{\mathbb{H}^+} |\nabla_H u|^p d\xi \geq -(p-1)(|\beta|^{\frac{p}{p-1}} + \beta) \int_{\mathbb{H}^+} \frac{\mathcal{W}(\xi)^p}{dist(\xi,\partial\mathbb{H}^+)^p} |u|^p d\xi.$$

Now we optimise by differentiating the above inequality with respect to β, so that we have

$$\frac{p}{p-1}|\beta|^{\frac{1}{p-1}} + 1 = 0,$$

which leads to

$$\beta = -\left(\frac{p-1}{p}\right)^{p-1}.$$

Using this value of β, we arrive at

$$\int_{\mathbb{H}^+} |\nabla_H u|^p d\xi \geq \left(\frac{p-1}{p}\right)^p \int_{\mathbb{H}^+} \frac{\mathcal{W}(\xi)^p}{dist(\xi,\partial\mathbb{H}^+)^p} |u|^p d\xi.$$

We have finished the proof of Corollary 2.5.1.

2.6 Geometric Hardy-Sobolev inequalities

In this section, we present the geometric Hardy-Sobolev inequality in the half space on the Heisenberg group.

Lemma 2.6.1 *Let* $\mathbb{H}^+$ *be a half-space of the Heisenberg group* $\mathbb{H}^n$*. Then for* $p \geq 2$*, there exists a constant* $C_p > 0$ *such that*

$$E_p[u] = \int_{\mathbb{H}^+} |\nabla_H u|^p d\xi - \left(\frac{p-1}{p}\right)^p \int_{\mathbb{H}^+} \frac{|\nabla_H dist(\xi,\partial\mathbb{H}^+)|^p}{dist(\xi,\partial\mathbb{H}^+)^p} |u|^p d\xi$$
$$\geq C_p \int_{\mathbb{H}^+} |dist(\xi,\partial\mathbb{H}^+)|^{p-1} |\nabla_H v|^p d\xi, \tag{2.36}$$

for all $u \in C_0^\infty(\mathbb{H}^+)$*, where* $dist(\xi,\partial\mathbb{H}^+) := \langle \xi, \nu\rangle - d$ *is the distance from* ξ *to the boundary,* $C_p = (2^{p-1}-1)^{-1}$*, and* $u(\xi) = dist(\xi,\partial\mathbb{H}^+)^{\frac{p-1}{p}} v(\xi)$.

The Euclidean version of such a lower estimate to the Hardy inequality was established by Barbaris, Filippas and Tertikas [43].

Proof of Lemma 2.6.1. Let us begin by recalling once again the angle function, denoted by $\mathcal{W}$,

$$|\nabla_H dist(\xi,\partial\mathbb{H}^+)|^p = |(X_1\langle \xi,\nu\rangle, \dots, X_n\langle \xi,\nu\rangle, Y_1\langle \xi,\nu\rangle, \dots, Y_n\langle \xi,\nu\rangle)|^p$$
$$= \left(\sum_{i=1}^n \langle X_i(\xi),\nu\rangle^2 + \langle Y_i(\xi),\nu\rangle^2\right)^{\frac{p}{2}}$$
$$= \mathcal{W}(\xi)^p. \tag{2.37}$$

Note that $X_i\langle \xi,\nu\rangle$ is equal to $\langle X_i(\xi),\nu\rangle$, see the proof of Theorem 2.5.1. This expression $|\nabla_H dist(\xi,\partial\mathbb{H}^+)|^p = \mathcal{W}(\xi)^p$ will be used later. For now we will estimate the following form

$$E_p[u] := \int_{\mathbb{H}^+} |\nabla_H u|^p d\xi - \left(\frac{p-1}{p}\right)^p \int_{\mathbb{H}^+} \frac{\mathcal{W}(\xi)^p}{dist(\xi,\partial\mathbb{H}^+)^p} |u|^p d\xi. \tag{2.38}$$

To estimate this, we introduce the following ground transform

$$u(\xi) = dist(\xi,\partial\mathbb{H}^+)^{\frac{p-1}{p}} v(\xi). \tag{2.39}$$

By inserting it into (2.38) and using (2.37), we have

$$E_p[u] = \int_{\mathbb{H}^+} \left| \frac{p-1}{p} dist(\xi,\partial\mathbb{H}^+)^{-\frac{1}{p}} \nabla_H dist(\xi,\partial\mathbb{H}^+) v + \right.$$

$dist(\xi, \partial\mathbb{H}^+)^{\frac{p-1}{p}} \nabla_H v \Big|^p d\xi$

$$- \left(\frac{p-1}{p}\right)^p \int_{\mathbb{H}^+} \frac{\mathcal{W}(\xi)^p}{dist(\xi,\partial\mathbb{H}^+)^p} |dist(\xi, \partial\mathbb{H}^+)^{\frac{p-1}{p}} v|^p d\xi$$

$$\geq \int_{\mathbb{H}^+} \Big| \frac{p-1}{p} dist(\xi, \partial\mathbb{H}^+)^{-\frac{1}{p}} v +$$

$$dist(\xi, \partial\mathbb{H}^+)^{\frac{p-1}{p}} \frac{\nabla_H v}{\nabla_H dist(\xi,\partial\mathbb{H}^+)} \Big|^p |\mathcal{W}(\xi)|^p$$

$$- \Big| \frac{p-1}{p} dist(\xi, \partial\mathbb{H}^+)^{-\frac{1}{p}} v \Big|^p |\mathcal{W}(\xi)|^p d\xi.$$

Then for $p \geq 2$ and $A, B \in \mathbb{R}^n$ we have that

$$|A+B|^p - |A|^p \geq C_p |B|^p + p|A|^{p-2} A \cdot B,$$

where $C_p = (2^{p-1}-1)^{-1}$. By taking

$$A := \frac{p-1}{p} dist(\xi, \partial\mathbb{H}^+)^{-\frac{1}{p}} v \quad \text{and} \quad B := dist(\xi, \partial\mathbb{H}^+)^{\frac{p-1}{p}} \frac{\nabla_H v}{\nabla_H dist(\xi, \partial\mathbb{H}^+)},$$

then we have the following lower estimate

$$E_p[u] \geq \int_{\mathbb{H}^+} |\mathcal{W}(\xi)|^p (|A+B|^p - |A|^p) d\xi$$

$$\geq C_p \int_{\mathbb{H}^+} dist(\xi, \partial\mathbb{H}^+)^{p-1} |\nabla_H v|^p \frac{\mathcal{W}(\xi)^p}{|\nabla_H dist(\xi, \partial\mathbb{H}^+)|^p} d\xi$$

$$+ \left(\frac{p-1}{p}\right)^{p-1} \int_{\mathbb{H}^+} |\mathcal{W}(\xi)|^p |\nabla_H dist(\xi, \partial\mathbb{H}^+)|^{p-2} (\nabla_H dist(\xi, \partial\mathbb{H}^+) \cdot \nabla_H |v|^p) d\xi$$

$$\geq C_p \int_{\mathbb{H}^+} dist(\xi, \partial\mathbb{H}^+)^{p-1} |\nabla_H v|^p d\xi.$$

In the last line we have used (2.37) and we dropped the last term on the right-hand side. This completes the proof of Lemma 2.6.1.

Now we are ready to obtain the geometric Hardy-Sobolev inequality in the half-space on the Heisenberg group $\mathbb{H}^n$.

Theorem 2.6.2 *Let $\mathbb{H}^+$ be a half-space of the Heisenberg group $\mathbb{H}^n$. Then for every function $u \in C_0^\infty(\mathbb{H}^+)$ and $2 \leq p < Q$ with $Q = 2n+1$, there exists some $C > 0$ such that we have*

$$\left(\int_{\mathbb{H}^+} |\nabla_H u|^p d\xi - \left(\frac{p-1}{p}\right)^p \int_{\mathbb{H}^+} \frac{\mathcal{W}(\xi)^p}{dist(\xi,\partial\mathbb{H}^+)^p} |u|^p d\xi\right)^{\frac{1}{p}} \geq C \left(\int_{\mathbb{H}^+} |u|^{p^*} d\xi\right)^{\frac{1}{p^*}}, \tag{2.40}$$

where $p^* := Qp/(Q-p)$ and $dist(\xi, \partial\mathbb{H}^+) := \langle \xi, \nu \rangle - d$ is the distance from ξ to *the boundary. Note that for $p = 2$ we have the Hardy-Sobolev-Maz'ya inequality in the following form*

$$\left(\int_{\mathbb{H}^+} |\nabla_H u|^2 d\xi - \frac{1}{4}\int_{\mathbb{H}^+} \frac{\mathcal{W}(\xi)^2}{dist(\xi,\partial\mathbb{H}^+)^2}|u|^2 d\xi\right)^{\frac{1}{2}} \geq C\left(\int_{\mathbb{H}^+} |u|^{2^*} d\xi\right)^{\frac{1}{2^*}}, \qquad (2.41)$$

where $2^* := 2Q/(Q-2)$.

P r o o f o f T h e o r e m 2 . 6 . 2 . Our key ingredient of proving the Hardy-Sobolev inequality in the half-space of $\mathbb{H}^n$ is the L^1-Sobolev inequality, or the Gagliardo-Nirenberg inequality. It has been established on the Heisenberg group by Baldi, Franchi, Pansu in [44].

The L^1-Sobolev inequality on the Heisenberg group follows in the form

$$c\left(\int_{\mathbb{H}^n} |g|^{\frac{Q}{Q-1}} d\xi\right)^{\frac{Q-1}{Q}} \leq \int_{\mathbb{H}^n} |\nabla_H g| d\xi,$$

for some $c > 0$, for every function $g \in W^{1,1}(\mathbb{H}^n)$. Now let us set $g = |u|^{p^*(1-1/Q)}$, then we obtain

$$\begin{aligned} c\left(\int_{\mathbb{H}^+} |u|^{p^*} d\xi\right)^{\frac{Q-1}{Q}} &\leq \left|\frac{p(Q-1)}{Q-p}\right| \int_{\mathbb{H}^+} |u|^{\frac{Q(p-1)}{Q-p}} |\nabla_H |u|| d\xi, \\ &\leq \left|\frac{p(Q-1)}{Q-p}\right| \int_{\mathbb{H}^+} |u|^{\frac{Qp}{Q-p}\frac{(p-1)}{p}} |\nabla_H u| d\xi, \\ &= \left|\frac{p(Q-1)}{Q-p}\right| \int_{\mathbb{H}^+} |u|^{p^*(1-1/p)} |\nabla_H u| d\xi. \end{aligned}$$

We have used $|\nabla_H|u|| \leq |\nabla_H u|$. Then we arrive at

$$C_1\left(\int_{\mathbb{H}^+} |u|^{p^*} d\xi\right)^{\frac{Q-1}{Q}} \leq \int_{\mathbb{H}^+} |u|^{p^*(1-1/p)} |\nabla_H u| d\xi, \qquad (2.42)$$

where $C_1 := c\left|\frac{Q-p}{p(Q-1)}\right| > 0$. Let us estimate the right-hand side of inequality (2.42). Again we use a ground transform $u(\xi) = dist(\xi,\partial\mathbb{H}^+)^{\frac{p-1}{p}} v(\xi)$ which leads to

$$\begin{aligned} &\int_{\mathbb{H}^+} |u|^{p^*(1-1/p)} |\nabla_H u| d\xi \\ &= \int_{\mathbb{H}^+} |u|^{p^*(1-1/p)} \Big| dist(\xi,\partial\mathbb{H}^+)^{\frac{p-1}{p}} \nabla_H v \\ &\qquad + \frac{p-1}{p} dist(\xi,\partial\mathbb{H}^+)^{-1/p} \nabla_H dist(\xi,\partial\mathbb{H}^+) v \Big| d\xi \end{aligned}$$

$$\leq \int_{\mathbb{H}^+} |u|^{p^*(1-1/p)} dist(\xi,\partial\mathbb{H}^+)^{\frac{p-1}{p}} |\nabla_H v| d\xi$$

$$+\frac{p-1}{p}\int_{\mathbb{H}^+} dist(\xi,\partial\mathbb{H}^+)^{p^*(1-1/p)^2-1/p} |\nabla_H dist(\xi,\partial\mathbb{H}^+)| |v|^{p^*(1-1/p)+1} d\xi$$

$$= I_1 + \frac{p-1}{p} I_2.$$

In the last line we have denoted two integrals by I_1 and I_2, respectively. Also, for simplification we denote $\alpha := p^*(1-1/p)^2 + 1 - 1/p$. First, we estimate I_2 using integrations by parts

$$\begin{aligned}
I_2 &= \int_{\mathbb{H}^+} dist(\xi,\partial\mathbb{H}^+)^{\alpha-1} |\nabla_H dist(\xi,\partial\mathbb{H}^+)| |v|^{\alpha p/(p-1)} d\xi \\
&= \frac{1}{\alpha}\int_{\mathbb{H}^+} \langle \nabla_H dist(\xi,\partial\mathbb{H}^+)^{\alpha}, \nabla_H dist(\xi,\partial\mathbb{H}^+)\rangle \frac{|v|^{\alpha p/(p-1)}}{|\nabla_H dist(\xi,\partial\mathbb{H}^+)|} d\xi \\
&= -\frac{1}{\alpha}\int_{\mathbb{H}^+} dist(\xi,\partial\mathbb{H}^+)^{\alpha} \nabla_H \left(\frac{\nabla_H dist(\xi,\partial\mathbb{H}^+)|v|^{\alpha p/(p-1)}}{|\nabla_H dist(\xi,\partial\mathbb{H}^+)|}\right) d\xi \\
&= -\frac{1}{\alpha}\int_{\mathbb{H}^+} dist(\xi,\partial\mathbb{H}^+)^{\alpha} \times \\
&\left(\frac{\nabla_H dist(\xi,\partial\mathbb{H}^+)\nabla_H|v|^{\alpha p/(p-1)}}{|\nabla_H dist(\xi,\partial\mathbb{H}^+)|} - \frac{\langle \nabla_H dist(\xi,\partial\mathbb{H}^+), \nabla_H|\nabla_H dist(\xi,\partial\mathbb{H}^+)|\rangle |v|^{\alpha p/(p-1)}}{|\nabla_H dist(\xi,\partial\mathbb{H}^+)|^2}\right) d\xi \\
&= -\frac{1}{\alpha}\int_{\mathbb{H}^+} dist(\xi,\partial\mathbb{H}^+)^{\alpha} \frac{\nabla_H dist(\xi,\partial\mathbb{H}^+)\nabla_H|v|^{\alpha p/(p-1)}}{|\nabla_H dist(\xi,\partial\mathbb{H}^+)|} d\xi \\
&\leq -\frac{p}{p-1}\int_{\mathbb{H}^+} dist(\xi,\partial\mathbb{H}^+)^{\alpha} |v|^{\alpha p/(p-1)-1} |\nabla_H v| d\xi \\
&= -\frac{p}{p-1}\int_{\mathbb{H}^+} |u|^{p^*(1-1/p)} dist(\xi,\partial\mathbb{H}^+)^{\frac{p-1}{p}} |\nabla_H v| d\xi \\
&\leq \frac{p}{p-1} I_1.
\end{aligned}$$

We have used $|\nabla_H|u|| \leq |\nabla_H u|$, and

$$\mathcal{L} dist(\xi,\partial\mathbb{H}^+) = \sum_{i=1}^{n} X_i\langle X_i(\xi),\nu\rangle + Y_i\langle Y_i(\xi),\nu\rangle = 0,$$

since

$$\langle X_i(\xi),\nu\rangle = \nu_{x,i} + 2y_i\nu_t, \langle Y_i(\xi),\nu\rangle = \nu_{y,i} - 2x_i\nu_t,$$

$$X_i\langle X_i(\xi),\nu\rangle = 0, Y_i\langle Y_i(\xi),\nu\rangle = 0,$$

$$Y_i\langle X_i(\xi),\nu\rangle = 2\nu_t, X_i\langle Y_i(\xi),\nu\rangle = -2\nu_t,$$

where $\xi := (x,y,t)$ with $x,y \in \mathbb{R}^n$ and $t \in \mathbb{R}$, $\nu := (\nu_x,\nu_y,\nu_t)$ with $\nu_x := (\nu_{x,1},\dots\nu_{x,n})$ and $\nu_y := (\nu_{y,1},\dots\nu_{y,n})$.

Also we have

$$\langle \nabla_H dist(\xi, \partial\mathbb{H}^+), \nabla_H |\nabla_H dist(\xi, \partial\mathbb{H}^+)|\rangle = \frac{2\nu_t}{|\nabla_H dist(\xi,\partial\mathbb{H}^+)|}$$
$$(\underbrace{(2x_1\nu_t - \nu_{y,1})(\nu_{x,1} + 2y_1\nu_t) + \cdots + (2x_n\nu_t - \nu_{y,n})(\nu_{x,n} + 2y_n\nu_t)}_{n}$$
$$+ \underbrace{(\nu_{y,1} - 2x_1\nu_t)(\nu_{x,1} + 2y_1\nu_t) + \cdots + (\nu_{y,n} - 2x_n\nu_t)(\nu_{x,n} + 2y_n\nu_t)}_{n}) = 0,$$

since

$$\nabla_H |\nabla_H dist(\xi, \partial\mathbb{H}^+)| = (X_1 |\nabla_H dist(\xi, \partial\mathbb{H}^+)|, \dots, X_n |\nabla_H dist(\xi, \partial\mathbb{H}^+)|,$$
$$Y_1 |\nabla_H dist(\xi, \partial\mathbb{H}^+)|, \dots, Y_n |\nabla_H dist(\xi, \partial\mathbb{H}^+)|)$$
$$=$$
$$\frac{2\nu_t}{|\nabla_H dist(\xi,\partial\mathbb{H}^+)|} (\underbrace{2x_1\nu_t - \nu_{y,1}, \dots, 2x_n\nu_t - \nu_{y,n}}_{n}, \underbrace{\nu_{x,1} + 2y_1\nu_t, \dots, \nu_{x,n} + 2y_n\nu_t}_{n}),$$

and

$$\nabla_H dist(\xi, \partial\mathbb{H}^+) = (\underbrace{\nu_{x,1} + 2y_1\nu_t, \dots, \nu_{x,n} + 2y_n\nu_t}_{n}, \underbrace{\nu_{y,1} - 2x_1\nu_t, \dots, \nu_{y,n} - 2x_n\nu_t}_{n}).$$

As we see that integral I_2 can be estimated by integral I_1. From this estimation we know that

$$\int_{\mathbb{H}^+} |u|^{p^*(1-1/p)} |\nabla_H u| d\xi \leq 2I_1. \tag{2.43}$$

Now it comes to estimate I_1 by using the Hölder inequality

$$I_1 = \int_{\mathbb{H}^+} \left\{|u|^{p^*(1-1/p)}\right\}\left\{dist(\xi, \partial\mathbb{H}^+)^{(p-1)/p} |\nabla_H v|\right\} d\xi$$
$$\leq \left(\int_{\mathbb{H}^+} |u|^{p^*} d\xi\right)^{1-1/p} \left(\int_{\mathbb{H}^+} dist(\xi, \partial\mathbb{H}^+)^{p-1} |\nabla_H v|^p d\xi\right)^{1/p}$$
$$\leq C_p^{-1/p} \left(\int_{\mathbb{H}^+} |u|^{p^*} d\xi\right)^{1-1/p} \left(\int_{\mathbb{H}^+} |\nabla_H u|^p d\xi\right.$$
$$\left. - \left(\frac{p-1}{p}\right)^p \int_{\mathbb{H}^+} \frac{\mathcal{W}(\xi)^p}{dist(\xi, \partial\mathbb{H}^+)^p} |u|^p d\xi\right)^{1/p}.$$

In the last line we have used Lemma 2.6.1. Inserting the estimate of I_1 in (2.43), we arrive at

$$\int_{\mathbb{H}^+} |u|^{p^*(1-1/p)} |\nabla_H u| d\xi \leq$$

$$2C_p^{-1/p}\left(\int_{\mathbb{H}^+}|u|^{p^*}d\xi\right)^{1-1/p}\left(\int_{\mathbb{H}^+}|\nabla_H u|^p d\xi\right.$$
$$\left.-\left(\frac{p-1}{p}\right)^p\int_{\mathbb{H}^+}\frac{\mathcal{W}(\xi)^p}{dist(\xi,\partial\mathbb{H}^+)^p}|u|^p d\xi\right)^{\frac{1}{p}}.$$

Plugging the above estimate in (2.42), we have

$$C_1\left(\int_{\mathbb{H}^+}|u|^{p^*}d\xi\right)^{\frac{Q-1}{Q}}\leq 2C_p^{-1/p}\left(\int_{\mathbb{H}^+}|u|^{p^*}d\xi\right)^{\frac{p-1}{p}}$$
$$\left(\int_{\mathbb{H}^+}|\nabla_H u|^p d\xi-\left(\frac{p-1}{p}\right)^p\int_{\mathbb{H}^+}\frac{\mathcal{W}(\xi)^p}{dist(\xi,\partial\mathbb{H}^+)^p}|u|^p d\xi\right)^{\frac{1}{p}}.$$

By collecting terms, we finish the proof of Theorem 2.6.2.

Let us demonstrate our result in a particular case when $p=2$:

Corollary 2.6.4 *Let* $\mathbb{H}^+:=\{\xi=(x,y,t)\in\mathbb{H}^n|\ t>0\}$ *be a half-space of the Heisenberg group* $\mathbb{H}^n$. *Then for every function* $u\in C_0^\infty(\mathbb{H}^+)$ *taking* $d=0$ *we have*

$$\left(\int_{\mathbb{H}^+}|\nabla_H u|^2 d\xi-\int_{\mathbb{H}^+}\frac{|x|^2+|y|^2}{t^2}|u|^2 d\xi\right)^{\frac{1}{2}}\geq C\left(\int_{\mathbb{H}^+}|u|^{2^*}d\xi\right)^{\frac{1}{2^*}}, \qquad (2.44)$$

where $2^*:=2Q/(Q-2)$, $Q=2n+2$, *with* $C>0$ *independent of* u.

Proof of Corollary 2.6.4. We have the following left-invariant vector fields

$$X_i=\frac{\partial}{\partial x_i}+2y_i\frac{\partial}{\partial t}\quad\text{and}\quad Y_i=\frac{\partial}{\partial y_i}-2x_i\frac{\partial}{\partial t},$$

with the commutator

$$[X_i,Y_i]=-4\frac{\partial}{\partial t}.$$

Then for $\xi=(x_1,\ldots,x_n,y_1,\ldots,y_n,t)$ and $\nu=(\overbrace{0,\ldots,0}^{n},\overbrace{0,\ldots,0}^{n},1)$, we get

$$\langle X_i(\xi),\nu\rangle=2y_i\quad\text{and}\quad\langle Y_i(\xi),\nu\rangle=-2x_i,$$

where

$$X_i(\xi)=(\overbrace{0,\ldots,1}^{i},\ldots,0,\overbrace{0,\ldots,0}^{n},2y_i),$$

$$Y_i(\xi) = (\overbrace{0,\ldots,0}^{n}, \overbrace{0,\ldots,1}^{i}, \ldots, 0,, -2x_i).$$

Thus, we arrive at

$$\frac{\mathcal{W}(\xi)^2}{dist(\xi,\partial\mathbb{H}^+)^2} = 4\frac{|x|^2+|y|^2}{t^2}. \tag{2.45}$$

Plugging the above expression into inequality (2.41) we obtain

$$\left(\int_{\mathbb{H}^+} |\nabla_H u|^2 d\xi - \int_{\mathbb{H}^+} \frac{|x|^2+|y|^2}{t^2}|u|^2 d\xi\right)^{\frac{1}{2}} \geq C\left(\int_{\mathbb{H}^+} |u|^{2^*} d\xi\right)^{\frac{1}{2^*}},$$

showing (2.44).

2.7 Geometric Hardy inequalities on starshaped sets

In order to present the results on the starshaped domains, let us recall the definition of starshaped sets in a Carnot group $\mathbb{G} = (\mathbb{R}^n, \circ, \delta_t)$ and related arguments.

Definition 2.7.1 [Starshapedness [45]] *Let $\Omega \subset \mathbb{G}$ be a C^1 domain containing the identity e. Then Ω is starshaped with respect to e if for every $x \in \partial\Omega$ one has*

$$\langle Z(x), n(x)\rangle \geq 0, \tag{2.46}$$

where n is the Riemannian outer normal to $\partial\Omega$.

When the strict inequality holds, then Ω is said to be strictly starshaped with respect to e. Here the vector fields Z are the infinitesimal generator of this group automorphism. This vector fields Z takes the form

$$Z = \sum_{i=1}^{N} x'_i \frac{\partial}{\partial x'_i} + 2\sum_{l=1}^{N_2} x_{2,l}\frac{\partial}{\partial x_{2,l}} + \cdots + r\sum_{l=1}^{N_r} x_{r,l}\frac{\partial}{\partial x_{r,l}}. \tag{2.47}$$

Then for $x' \in \mathbb{R}^N$ and $x^{(i)} \in \mathbb{R}^{N_i}$ with $i = 2, \ldots, r$ we have

$$Z(x) = (x', 2x^{(2)}, \cdots, rx^{(r)}), \tag{2.48}$$

and

$$\begin{aligned}\langle Z(x), n(x)\rangle &= x'n' + 2x^{(2)}n^{(2)} + \cdots + rx^{(r)}n^{(r)}\\ &= x'_1 n'_1 + \cdots + x'_N n'_N + 2(x_{2,1}n_{2,1} + \cdots + x_{2,N_2}n_{2,N_2})\\ &+ \cdots + r(x_{r,1}n_{r,1} + \cdots + x_{r,N_r}n_{r,N_r}),\end{aligned}$$

since $n(x) := (n', n^{(2)}, \ldots, n^{(r)})$ with $n' \in \mathbb{R}^N$ and $n^{(i)} \in \mathbb{R}^{N_i}$, $i = 2, \ldots, r$.

Based on the above arguments now we present the geometric Hardy inequalities on the starshaped sets for the sub-Laplacians.

Theorem 2.7.2 *Let Ω be a starshaped set on a Carnot group. Then for every $\gamma \in \mathbb{R}$ and $p > 1$ we have the following Hardy inequality*

$$\int_{\Omega} |\nabla_H f(x)|^p dx \geq -(p-1)(|\gamma|^{\frac{p}{p-1}} + \gamma) \int_{\Omega} \frac{|\nabla_H \langle Z(x), n(x)\rangle|^p}{|\langle Z(x), n(x)\rangle|^p} |f(x)|^p dx \quad (2.49)$$
$$+\gamma \int_{\Omega} \frac{\mathcal{L}_p(\langle Z(x), n(x)\rangle)}{|\langle Z(x), n(x)\rangle|^{p-1}} |f(x)|^p dx,$$

for every function $f \in C_0^{\infty}(\Omega)$.

Proof of Theorem 2.7.2. The approach to prove the main results is based on [46]. For a vector field $g \in C^{\infty}(\Omega)$ we compute

$$\int_{\Omega} \mathrm{d}iv_X g |f(x)|^p dx = -p \int_{\Omega} |f(x)|^{p-1} \langle g, \nabla_X f(x)\rangle dx$$
$$\leq p \left(\int_{\Omega} |\nabla_X f(x)|^p dx\right)^{\frac{1}{p}} \left(\int_{\Omega} |g|^{\frac{p}{p-1}} |f(x)|^p dx\right)^{\frac{p-1}{p}}$$
$$\leq \int_{\Omega} |\nabla_H f(x)|^p dx + (p-1) \int_{\Omega} |g|^{\frac{p}{p-1}} |f(x)|^p dx.$$

Here we have first used the divergence theorem, then we applied the Hölder inequality and the Young inequality. By rearranging the above expression, we arrive at

$$\int_{\Omega} |\nabla_X f(x)|^p dx \geq \int_{\Omega} (\mathrm{d}iv_X g - (p-1)|g|^{\frac{p}{p-1}}) |f(x)|^p dx. \quad (2.50)$$

A suitable choice of the vector field g in each special case is a key argument of our proofs. Let us set

$$g = \gamma \frac{|\nabla_H \langle Z(x), n(x)\rangle|^{p-2}}{|\langle Z(x), n(x)\rangle|^{p-1}} \nabla_H \langle Z(x), n(x)\rangle,$$

so that we have

$$|g|^{\frac{p}{p-1}} = |\gamma|^{\frac{p}{p-1}} \frac{|\nabla_H \langle Z(x), n(x)\rangle|^p}{|\langle Z(x), n(x)\rangle|^p}, \quad (2.51)$$

and

$$\mathrm{d}iv_H g = \gamma \frac{\mathcal{L}_p(\langle Z(x), n(x)\rangle)}{|\langle Z(x), n(x)\rangle|^{p-1}} - \gamma(p-1) \frac{|\nabla_H \langle Z(x), n(x)\rangle|^p}{|\langle Z(x), n(x)\rangle|^p}. \quad (2.52)$$

Plugging the above expressions (2.51) and (2.52) into inequality (2.50), we get

$$\int_{\Omega} |\nabla_H f(x)|^p dx \geq -(p-1)(|\gamma|^{\frac{p}{p-1}} + \gamma) \int_{\Omega} \frac{|\nabla_H \langle Z(x), n(x)\rangle|^p}{|\langle Z(x), n(x)\rangle|^p} |f(x)|^p dx$$
$$+\gamma \int_{\Omega} \frac{\mathcal{L}_p(\langle Z(x), n(x)\rangle)}{|\langle Z(x), n(x)\rangle|^{p-1}} |f(x)|^p dx,$$

which proves inequality (2.49).

Corollary 2.7.3 *Let* $\mathbb{H}^*$ *be a starshaped set on the Heisenberg group* $\mathbb{H}_1$. *Then for* $p > 1$, *we have the following Hardy inequality*

$$\int_{\mathbb{H}^*} |\nabla_H f(x)|^p dx \geq \left(\frac{p-1}{p}\right)^p \int_{\mathbb{H}^*} \frac{|(n_1+4x_2n_3, n_2-4x_1n_3)|^p}{|x_1n_1+x_2n_2+2x_3n_3|^p} |f(x)|^p dx, \quad (2.53)$$

for every function $f \in C_0^\infty(\mathbb{H}^*)$.

Proof of Corollary 2.7.3. We begin the proof of Corollary 2.7.3 by a simple computation such as

$$\langle Z(x), n(x)\rangle = x_1n_1 + x_2n_2 + 2x_3n_3,$$

$$\nabla_H \langle Z(x), n(x)\rangle = (n_1 + 4x_2n_3, n_2 - 4x_1n_3),$$

$$|\nabla_H \langle Z(x), n(x)\rangle|^p = ((n_1 + 4x_2n_3)^2 + (n_2 - 4x_1n_3)^2)^{p/2},$$

and

$$\mathcal{L}_p \langle Z(x), n(x)\rangle = \nabla_H \cdot (|\nabla_H \langle Z(x), n(x)\rangle|^{p-2} \nabla_H \langle Z(x), n(x)\rangle)$$

$$= X_1(|\nabla_H \langle Z(x), n(x)\rangle|^{p-2}(n_1 + 4x_2n_3))$$

$$+X_2(|\nabla_H \langle Z(x), n(x)\rangle|^{p-2}(n_2 - 4x_1n_3))$$
$$= -4(p-2)|\nabla_H \langle Z(x), n(x)\rangle|^{p-4}(n_1 + 4x_2n_3)(n_2 - 4x_1n_3)n_3$$

$$+4(p-2)|\nabla_H \langle Z(x), n(x)\rangle|^{p-4}(n_2 - 4x_1n_3)(n_1 + 2x_4n_3)n_3$$

$$= 0.$$

Plugging the above expressions into inequality (2.49) and maximising with respect to γ, we arrive at inequality (2.53) which proves Corollary 2.7.3.

Corollary 2.7.4 *Let* $\mathbb{E}^*$ *be a starshaped set on the Engel group* $\mathbb{E}$. *Then for every function* $f \in C_0^\infty(\mathbb{E}^*)$, $\gamma \in \mathbb{R}$ *and* $p = 2$, *we have*

$$\int_{\mathbb{E}^*} |\nabla_H f(x)|^2 dx \geq -(|\gamma|^2 + \gamma) \int_{\mathbb{E}^*} \frac{|\nabla_H \langle Z(x), n(x)\rangle|^2}{\langle Z(x), n(x)\rangle^2} |f(x)|^2 dx \quad (2.54)$$
$$+ \frac{\gamma}{2} \int_{\mathbb{E}^*} \frac{x_2n_4}{\langle Z(x), n(x)\rangle} |f(x)|^2 dx.$$

Proof of Corollary 2.7.4. We begin the proof of Corollary 2.7.4 by a simple computation such as

$$\langle Z(x), n(x)\rangle = x_1 n_1 + x_2 n_2 + 2x_3 n_3 + 3x_4 n_4,$$

$$\nabla_H \langle Z(x), n(x)\rangle = \left(n_1 - x_2 n_3 - \frac{3x_3 n_4}{2} - \frac{x_1 x_2 n_4}{4}, n_2 + x_1 n_3 + \frac{x_1^2 n_4}{4}\right),$$

$$|\nabla_H \langle Z(x), n(x)\rangle|^2 = \left(n_1 - x_2 n_3 - \frac{3x_3 n_4}{2} - \frac{x_1 x_2 n_4}{4}\right)^2 + \left(n_2 + x_1 n_3 + \frac{x_1^2 n_4}{4}\right)^2,$$

and

$$\begin{gathered}\mathcal{L}\langle Z(x), n(x)\rangle = \nabla_H \cdot \nabla_H \langle Z(x), n(x)\rangle \\ = X_1\left(n_1 - x_2 n_3 - \frac{3x_3 n_4}{2} - \frac{x_1 x_2 n_4}{4}\right) + X_2\left(n_2 + x_1 n_3 + \frac{x_1^2 n_4}{4}\right) \\ = \frac{x_2 n_4}{2}.\end{gathered}$$

Plugging the above expressions into inequality (2.49)

$$\begin{gathered}\int_{\mathbb{E}^*} |\nabla_H f(x)|^2 dx \geq -(|\gamma|^2 + \gamma)\int_{\mathbb{E}^*} \frac{|\nabla_H \langle Z(x), n(x)\rangle|^2}{\langle Z(x), n(x)\rangle^2} |f(x)|^2 dx \\ + \frac{\gamma}{2}\int_{\mathbb{E}^*} \frac{x_2 n_4}{\langle Z(x), n(x)\rangle} |f(x)|^2 dx,\end{gathered}$$

which proves Corollary 2.7.4.

3 HORIZONTAL HARDY AND RELLICH INEQUALITIES

In this chapter, we discuss versions of Hardy and Rellich type inequalities on the stratified groups with the Euclidean distance on the first stratum of the stratified group.

3.1 Horizontal anisotropic Hardy and Rellich inequalities

In this section, the anisotropic versions of horizontal Hardy and Rellich inequalities are discussed, where they appear in the analysis of anisotropic p-sub-Laplacians. To put the notions in perspective, we start by recalling the Euclidean counterparts of the appearing objects.

Let us recall the anisotropic Laplacian on $\mathbb{R}^N$ which is defined by

$$\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p_i-2} \frac{\partial u}{\partial x_i}\right), \tag{3.1}$$

for $p_i > 1$ where $i = 1, \dots, N$ [47]. Note that by taking $p_i = 2$ or $p_i = p = const$ in (3.1) we get the Laplacian and the pseudo-p-Laplacian, respectively. The anisotropic Laplacian has the theoretical importance not only in mathematics but also many practical applications in the natural sciences. There are several examples: it reflects anisotropic physical properties of some reinforced materials Lions [48] and Tang [49], as well as explains the dynamics of fluids in the anisotropic media when the conductivities of the media are different in each direction [50, 51]. It has also applications in image processing [52].

Here we present the horizontal anisotropic Picone type identity on a stratified group $\mathbb{G}$.

Lemma 3.1.1 *Let $\boldsymbol{\Omega} \subset \mathbb{G}$ be an open set, where $\mathbb{G}$ is a stratified group with $\boldsymbol{N}$ being the dimension of the first stratum. Let $\boldsymbol{u}, \boldsymbol{v}$ be differentiable a.e. in $\boldsymbol{\Omega}$, $\boldsymbol{v} > \mathbf{0}$ a.e. in $\boldsymbol{\Omega}$ and $\boldsymbol{u} \geq \mathbf{0}$, and denote*

$$R(u,v) := \sum_{i=1}^{N} |X_i u|^{p_i} - \sum_{i=1}^{N} X_i\left(\frac{u^{p_i}}{v^{p_i-1}}\right)|X_i v|^{p_i-2} X_i v, \tag{3.2}$$

$$\begin{aligned} L(u,v) := {} & \sum_{i=1}^{N} |X_i u|^{p_i} - \sum_{i=1}^{N} p_i \frac{u^{p_i-1}}{v^{p_i-1}} |X_i v|^{p_i-2} X_i v X_i u \\ & + \sum_{i=1}^{N} (p_i - 1) \frac{u^{p_i}}{v^{p_i}} |X_i v|^{p_i}, \end{aligned} \tag{3.3}$$

where $p_i > 1$, $i = 1, \dots, N$. Then

$$L(u,v) = R(u,v) \geq 0. \tag{3.4}$$

In addition, for simply connected Ω we have $L(u,v) = 0$ a.e. in Ω if and only if $u = cv$ a.e. in Ω with a positive constant c.

Remark 3.1.2 Note that the Euclidean case of Lemma 3.1.1 was obtained by Feng and Cui.

Note that the proof of Lemma 3.1.1 is based on the method of Allegretto and Huang [53] for the p-Laplacian.

Proof of Lemma 3.1.1. A direct computation gives

$$
\begin{aligned}
R(u,v) &= \sum_{i=1}^{N} |X_i u|^{p_i} - \sum_{i=1}^{N} X_i\left(\frac{u^{p_i}}{v^{p_i-1}}\right)|X_i v|^{p_i-2}X_i v \\
&= \sum_{i=1}^{N} |X_i u|^{p_i} - \sum_{i=1}^{N} \frac{p_i u^{p_i-1} X_i u v^{p_i-1} - u^{p_i}(p_i-1)v^{p_i-2}X_i v}{(v^{p_i-1})^2}|X_i v|^{p_i-2}X_i v \\
&= \sum_{i=1}^{N} |X_i u|^{p_i} - \sum_{i=1}^{N} p_i \frac{u^{p_i-1}}{v^{p_i-1}}|X_i v|^{p_i-2}X_i v X_i u + \sum_{i=1}^{N} (p_i - 1)\frac{u^{p_i}}{v^{p_i}}|X_i v|^{p_i} \\
&= L(u,v).
\end{aligned}
$$

This proves the equality in (3.4). Now we rewrite $L(u,v)$ to see $L(u,v) \geq 0$, that is,

$$
\begin{aligned}
L(u,v) &= \sum_{i=1}^{N} |X_i u|^{p_i} - \sum_{i=1}^{N} p_i \frac{u^{p_i-1}}{v^{p_i-1}}|X_i v|^{p_i-1}|X_i u| + \sum_{i=1}^{N} (p_i - 1)\frac{u^{p_i}}{v^{p_i}}|X_i v|^{p_i} \\
&+ \sum_{i=1}^{N} p_i \frac{u^{p_i-1}}{v^{p_i-1}}|X_i v|^{p_i-2}(|X_i v||X_i u| - X_i v X_i u) \\
&= S_1 + S_2,
\end{aligned}
$$

where we denote

$$
\begin{aligned}
S_1 := \sum_{i=1}^{N} p_i &\left[\frac{1}{p_i}|X_i u|^{p_i} + \frac{p_i-1}{p_i}\left(\left(\frac{u}{v}|X_i v|\right)^{p_i-1}\right)^{\frac{p_i}{p_i-1}}\right] \\
&- \sum_{i=1}^{N} p_i \frac{u^{p_i-1}}{v^{p_i-1}}|X_i v|^{p_i-1}|X_i u|,
\end{aligned}
$$

and

$$
S_2 := \sum_{i=1}^{N} p_i \frac{u^{p_i-1}}{v^{p_i-1}}|X_i v|^{p_i-2}(|X_i v||X_i u| - X_i v X_i u).
$$

We can see that $S_2 \geq 0$ due to $|X_i v||X_i u| \geq X_i v X_i u$. To check $S_1 \geq 0$ we need to use Young's inequality for $a \geq 0$ and $b \geq 0$

$$
ab \leq \frac{a^{p_i}}{p_i} + \frac{b^{q_i}}{q_i}, \tag{3.5}
$$

where $p_i > 1, q_i > 1$ and $\frac{1}{p_i} + \frac{1}{q_i} = 1$ for $i = 1, \ldots, N$. It holds if and only if $a^{p_i} = b^{q_i}$, i.e. if $a = b^{\frac{1}{p_i-1}}$. Let us take $a = |X_i u|$ and $b = \left(\frac{u}{v}|X_i v|\right)^{p_i-1}$ in (3.5) to get

$$p_i|X_iu|\left(\frac{u}{v}|X_iv|\right)^{p_i-1} \leq p_i\left[\frac{1}{p_i}|X_iu|^{p_i}+\frac{p_i-1}{p_i}\left(\left(\frac{u}{v}|X_iv|\right)^{p_i-1}\right)^{\frac{p_i}{p_i-1}}\right]. \quad (3.6)$$

From this we see that $S_1 \geq 0$ which proves that $L(u,v) = S_1 + S_2 \geq 0$. It is easy to see that $u = cv$ implies $R(u,v) = 0$. Now let us prove that $L(u,v) = 0$ implies $u = cv$. Due to $u(x) \geq 0$ and $L(u,v)(x_0) = 0$, $x_0 \in \Omega$, we consider the two cases $u(x_0) > 0$ and $u(x_0) = 0$.

For the case $u(x_0) > 0$ we conclude from $L(u,v)(x_0) = 0$ that $S_1 = 0$ and $S_2 = 0$. Then $S_1 = 0$ implies

$$|X_iu| = \frac{u}{v}|X_iv|, \quad i = 1, \ldots, N, \quad (3.7)$$

and $S_2 = 0$ implies

$$|X_iv||X_iu| - X_ivX_iu = 0, \quad i = 1, \ldots, N. \quad (3.8)$$

The combination of (3.7) and (3.8) gives

$$\frac{X_iu}{X_iv} = \frac{u}{v} = c, \quad \text{with} \quad c \neq 0, \quad i = 1, \ldots, N. \quad (3.9)$$

Let us denote $\Omega^* := \{x \in \Omega | u(x) = 0\}$. If $\Omega^* \neq \Omega$, then suppose that $x_0 \in \partial\Omega^*$. Then there exists a sequence $x_k \notin \Omega^*$ such that $x_k \to x_0$. In particular, $u(x_k) \neq 0$, and hence by the case 1 we have $u(x_k) = cv(x_k)$. Passing to the limit we get $u(x_0) = cv(x_0)$. Since $u(x_0) = 0$, $v(x_0) \neq 0$, we get that $c = 0$. But then by the case 1 again, since $u = cv$ and $u \neq 0$ in $\Omega \backslash \Omega^*$, it is impossible that $c = 0$. This contradiction implies that $\Omega^* = \Omega$.

This completes the proof of Lemma 3.1.1.

Also, we present the (second order) horizontal anisotropic Picone type identity.

Lemma 3.1.3 *Let $\Omega \subset \mathbb{G}$ be an open set, where $\mathbb{G}$ is a stratified group with N being the dimension of the first stratum. Let u, v be twice differentiable a.e. in Ω and satisfying the following conditions: $u \geq 0$, $v > 0$, $X_i^2 v < 0$ a.e. in Ω for $p_i > 1$, $i = 1, \ldots, N$. Then we have*

$$L_1(u,v) = R_1(u,v) \geq 0, \quad (3.10)$$

where

$$R_1(u,v) := \sum_{i=1}^{N} |X_i^2u|^{p_i} - \sum_{i=1}^{N} X_i^2\left(\frac{u^{p_i}}{v^{p_i-1}}\right)|X_i^2v|^{p_i-2}X_i^2v,$$

and

$$L_1(u,v) := \sum_{i=1}^{N} |X_i^2u|^{p_i} - \sum_{i=1}^{N} p_i\left(\frac{u}{v}\right)^{p_i-1} X_i^2uX_i^2v|X_i^2v|^{p_i-2}$$

$$+\sum_{i=1}^{N}(p_i-1)\left(\frac{u}{v}\right)^{p_i}|X_i^2v|^{p_i}$$
$$-\sum_{i=1}^{N}p_i(p_i-1)\frac{u^{p_i-2}}{v^{p_i-1}}|X_i^2v|^{p_i-2}X_i^2v\left(X_iu-\frac{u}{v}X_iv\right)^2.$$

Proof of Lemma 3.1.3. A direct computation gives

$$X_i^2\left(\frac{u^{p_i}}{v^{p_i-1}}\right)=X_i\left(p_i\frac{u^{p_i-1}}{v^{p_i-1}}X_iu-(p_i-1)\frac{u^{p_i}}{v^{p_i}}X_iv\right)$$
$$=p_i(p_i-1)\frac{u^{p_i-2}}{v^{p_i-2}}\left(\frac{(X_iu)v-u(X_iv)}{v^2}\right)X_iu+p_i\frac{u^{p_i-1}}{v^{p_i-1}}X_i^2u$$
$$-p_i(p_i-1)\frac{u^{p_i-1}}{v^{p_i-1}}\left(\frac{(X_iu)v-u(X_iv)}{v^2}\right)X_iv-(p_i-1)\frac{u^{p_i}}{v^{p_i}}X_i^2v$$
$$=p_i(p_i-1)\left(\frac{u^{p_i-2}}{v^{p_i-1}}|X_iu|^2-2\frac{u^{p_i-1}}{v^{p_i}}X_ivX_iu+\frac{u^{p_i}}{v^{p_i+1}}|X_iv|^2\right)$$
$$+p_i\frac{u^{p_i-1}}{v^{p_i-1}}X_i^2u-(p_i-1)\frac{u^{p_i}}{v^{p_i}}X_i^2v$$
$$=p_i(p_i-1)\frac{u^{p_i-2}}{v^{p_i-1}}\left(X_iu-\frac{u}{v}X_iv\right)^2+p_i\frac{u^{p_i-1}}{v^{p_i-1}}X_i^2u-(p_i-1)\frac{u^{p_i}}{v^{p_i}}X_i^2v,$$

which gives (3.10). By the Young inequality we have

$$\frac{u^{p_i-1}}{v^{p_i-1}}X_i^2uX_i^2v|X_i^2v|^{p_i-2}\leq\frac{|X_i^2u|^{p_i}}{p_i}+\frac{1}{q_i}\frac{u^{p_i}}{v^{p_i}}|X_i^2v|^{p_i},\quad i=1,\dots,N,$$

where $p_i>1, q_i>1$ $\frac{1}{p_i}+\frac{1}{q_i}=1$. Since $X_i^2v<0$ we arrive at

$$L_1(u,v)\geq\sum_{i=1}^{N}|X_i^2u|^{p_i}+\sum_{i=1}^{N}(p_i-1)\frac{u^{p_i}}{v^{p_i}}|X_i^2v|^{p_i}-$$
$$\sum_{i=1}^{N}p_i\left(\frac{|X_i^2u|^{p_i}}{p_i}+\frac{1}{q_i}\frac{u^{p_i}}{v^{p_i}}|X_i^2v|^{p_i}\right)$$
$$-\sum_{i=1}^{N}p_i(p_i-1)\frac{u^{p_i-2}}{v^{p_i-1}}|X_i^2v|^{p_i-2}X_i^2v\left|X_iu-\frac{u}{v}X_iv\right|^2$$
$$=\sum_{i=1}^{N}\left(p_i-1-\frac{p_i}{q_i}\right)\frac{u^{p_i}}{v^{p_i}}|X_i^2v|^{p_i}$$
$$-\sum_{i=1}^{N}p_i(p_i-1)\frac{u^{p_i-2}}{v^{p_i-1}}|X_i^2v|^{p_i-2}X_i^2v\left|X_iu-\frac{u}{v}X_iv\right|^2\geq 0.$$

This completes the proof of Lemma 3.1.3.

As a consequence of the horizontal anisotropic Picone type identity, we present the horizontal Hardy type inequality for the anisotropic sub-Laplacian on $\mathbb{G}$. Let us recall that $x=(x',x'')\in\mathbb{G}$ with x' being in the first stratum of $\mathbb{G}$.

Theorem 3.1.4 *Let $\Omega\subset\mathbb{G}\backslash\{x'=0\}$ be an open set, where $\mathbb{G}$ is a stratified group with N being the dimension of the first stratum. Then we have*

$$\sum_{i=1}^{N}\int_\Omega|X_iu|^{p_i}dx\geq\sum_{i=1}^{N}\left(\frac{p_i-1}{p_i}\right)^{p_i}\int_\Omega\frac{|u|^{p_i}}{|x'_i|^{p_i}}dx,\qquad(3.11)$$

for all $u \in C^1(\Omega)$ *and where* $1 < p_i < N$ *for* $i = 1, \dots, N$.

Before we start the proof of Theorem 3.1.4, let us establish the following Lemma 3.1.5.

Lemma 3.1.5 *Let* $\Omega \subset \mathbb{G}$ *be an open set, where* $\mathbb{G}$ *is a stratified group with* N *being the dimension of the first stratum. Let constants* $K_i > 0$ *and functions* $H_i(x)$ *with* $i = 1, \dots, N$, *be such that for an a.e. differentiable function* v, *such that* $v > 0$ *a.e. in* Ω, *we have*

$$-X_i(|X_i v|^{p_i-2} X_i v) \geq K_i H_i(x) v^{p_i-1}, \quad i = 1, \dots, N. \tag{3.12}$$

Then, for all nonnegative functions $u \in C^1(\Omega)$ *we have*

$$\sum_{i=1}^N \int_\Omega |X_i u|^{p_i} dx \geq \sum_{i=1}^N K_i \int_\Omega H_i(x) u^{p_i} dx. \tag{3.13}$$

Proof of Lemma 3.1.5. In view of (3.4) and (3.12) we have

$$0 \leq \int_\Omega L(u,v)dx = \int_\Omega R(u,v)dx$$

$$= \sum_{i=1}^N \int_\Omega |X_i u|^{p_i} dx - \sum_{i=1}^N \int_\Omega X_i \left(\frac{u^{p_i}}{v^{p_i-1}}\right) |X_i v|^{p_i-2} X_i v dx$$

$$= \sum_{i=1}^N \int_\Omega |X_i u|^{p_i} dx + \sum_{i=1}^N \int_\Omega \frac{u^{p_i}}{v^{p_i-1}} X_i(|X_i v|^{p_i-2} X_i v) dx$$

$$\leq \sum_{i=1}^N \int_\Omega |X_i u|^{p_i} dx - \sum_{i=1}^N K_i \int_\Omega H_i(x) u^{p_i} dx.$$

This completes the proof of Lemma 3.1.5.

Proof of Theorem 3.1.4. Before using Lemma 3.1.5, we shall introduce the auxiliary function

$$v := \prod_{j=1}^N |x'_j|^{\alpha_j} = |x'_i|^{\alpha_i} V_i, \tag{3.14}$$

where $V_i = \prod_{j=1, j\neq i}^N |x_{j'}|^{\alpha_j}$ and $\alpha_j = \frac{p_j - 1}{p_j}$. Then we have

$$X_i v = \alpha_i V_i |x_{i'}|^{\alpha_i - 2} x'_i,$$
$$\left|X_i v\right|^{p_i-2} = \alpha_i^{p_i-2} V_i^{p_i-2} \left|x_{i'}\right|^{\alpha_i p_i - 2\alpha_i - p_i + 2},$$

$$|X_i v|^{p_i-2} X_i v = \alpha_i^{p_i-1} V_i^{p_i-1} |x_{i'}|^{\alpha_i p_i - \alpha_i - p_i} x_{i'}.$$

Consequently, we also have

$$-X_i(|X_i v|^{p_i-2} X_i v) = \left(\frac{p_i - 1}{p_i}\right)^{p_i} \frac{v^{p_i-1}}{|x_{i'}|^{p_i}}. \tag{3.15}$$

To complete the proof of Theorem 3.1.4, we choose $K_i = \left(\frac{p_i - 1}{p_i}\right)^{p_i}$ and $H_i(x) = \frac{1}{|x_{i'}|^{p_i}}$, and use Lemma 3.1.5.

Now we are ready to prove the anisotropic Rellich type inequality on $\mathbb{G}$.

Theorem 3.1.6 *Let $\Omega \subset \mathbb{G}\backslash\{x' = 0\}$ be an open set, where $\mathbb{G}$ is a stratified group with N being the dimension of the first stratum. Then for a function $u \geq 0$, $u \in C^2(\Omega)$, and $2 < \alpha_i < N - 2$ we have the following inequality*

$$\sum_{i=1}^{N} \int_{\Omega} |X_i^2 u|^{p_i} dx \geq \sum_{i=1}^{N} C_i(\alpha_i, p_i) \int_{\Omega} \frac{|u|^{p_i}}{|x_{i'}|^{2p_i}} dx, \tag{3.16}$$

where $1 < p_i < N$ for $i = 1, \dots, N$, and

$$C_i(\alpha_i, p_i) = (\alpha_i(\alpha_i - 1))^{p_i - 1}(\alpha_i p_i - 2p_i - \alpha_i + 2)(\alpha_i p_i - 2p_i - \alpha_i + 1).$$

Proof of Theorem 3.1.6. We introduce the auxiliary function

$$v := \prod_{j=1}^{N} |x_{j'}|^{\alpha_j} = |x_{i'}|^{\alpha_i} V_i,$$

we choose α_j later, and let $V_i = \prod_{j=1, j\neq i}^{N} |x_{j'}|^{\alpha_j}$. Then we have

$$X_i^2 v = X_i(\alpha_i V_i |x_{i'}|^{\alpha_i - 2} x_{i'}) = \alpha_i(\alpha_i - 1) V_i |x_{i'}|^{\alpha_i - 2},$$

$$|X_i^2 v|^{p_i - 2} = (\alpha_i(\alpha_i - 1))^{p_i - 2} V_i^{p_i - 2} |x_{i'}|^{\alpha_i p_i - 2p_i - 2\alpha_i + 4},$$

$$|X_i^2 v|^{p_i - 2} X_i^2 v = (\alpha_i(\alpha_i - 1))^{p_i - 1} V_i^{p_i - 1} |x_{i'}|^{\alpha_i p_i - 2p_i - \alpha_i + 2}.$$

Consequently, we obtain

$$X_i^2(|X_i^2 v|^{p_i - 2} X_i^2 v) = (\alpha_i(\alpha_i - 1))^{p_i - 1} V_i^{p_i - 1} X_i^2(|x_{i'}|^{\alpha_i p_i - 2p_i - \alpha_i + 2})$$

$$= (\alpha_i(\alpha_i - 1))^{p_i - 1}(\alpha_i p_i - 2p_i - \alpha_i + 2) V_i^{p_i - 1} X_i(|x_{i'}|^{\alpha_i p_i - 2p_i - \alpha_i} x_{i'})$$

$$= (\alpha_i(\alpha_i - 1))^{p_i - 1}(\alpha_i p_i - 2p_i - \alpha_i + 2)(\alpha_i p_i - 2p_i - \alpha_i + 1)$$

$$\times V_i^{p_i - 1} |x_{i'}|^{\alpha_i(p_i - 1) - 2p_i}.$$

Thus, for twice differentiable function $v > 0$ a.e. in Ω with $X_i^2 v < 0$ we have

$$X_i^2(|X_i^2|^{p_i - 2} X_i^2 v) = C_i(\alpha_i, p_i) \frac{v^{p_i - 1}}{|x'_i|^{2p_i}} \tag{3.17}$$

a.e. in Ω. Using (3.17) we compute

$$0 \leq \int_\Omega L_1(u,v)dx = \int_\Omega R_1(u,v)dx$$

$$= \sum_{i=1}^N \int_\Omega |X_i^2 u|^{p_i} dx - \sum_{i=1}^N \int_\Omega X_i^2 \left(\frac{u^{p_i}}{v^{p_i-1}}\right) |X_i^2 v|^{p_i-2} X_i^2 v dx$$

$$= \sum_{i=1}^N \int_\Omega |X_i^2 u|^{p_i} dx - \sum_{i=1}^N \int_\Omega \frac{u^{p_i}}{v^{p_i-1}} X_i^2 \left(|X_i^2 v|^{p_i-2} X_i^2 v\right) dx$$

$$= \sum_{i=1}^N \int_\Omega |X_i^2 u|^{p_i} dx - \sum_{i=1}^N C_i(\alpha_i, p_i) \int_\Omega \frac{|u|^{p_i}}{|x_{i'}|^{2p_i}} dx.$$

The proof of Theorem 3.1.6 is complete.

3.2 Hardy type inequalities with multiple singularities

In this section, the analogue of the Hardy inequality with multiple singularities are presented on a stratified group. The singularities are represented by a family $\{a_k\}_{k=1}^m \in \mathbb{G}$, where we write $a_{\square} = (a_k', a_k'')$, with a_k' being in the first stratum of $\mathbb{G}$. We can also write $a_k' = (a_{k1}', \dots, a_{kN}')$. From [29, Proposition 3.1.24] it follows that $(xa_k^{-1})' = x' - a_k'$.

Theorem 3.2.1 *Let $\Omega \subset \mathbb{G}$ be an open set, where $\mathbb{G}$ is a stratified group with N being the dimension of the first stratum. Let $N \geq 3$, $x = (x', x'') \in \mathbb{G}$ with $x' = (x'_1, \dots, x'_N)$ being in the first stratum of $\mathbb{G}$, and let $a_k \in \mathbb{G}$, $k = 1, \dots, m$, be the singularities. Then we have*

$$\int_\Omega |\nabla_{\mathbb{G}} u|^2 dx \geq \left(\frac{N-2}{2}\right)^2 \int_\Omega \frac{\sum_{j=1}^N \left|\sum_{k=1}^m \frac{(xa_k^{-1})_{j'}}{|(xa_k^{-1})'|^N}\right|^2}{\left(\sum_{k=1}^m \frac{1}{|(xa_k^{-1})'|^{N-2}}\right)^2} |u|^2 dx, \tag{3.18}$$

for all $u \in C_0^\infty(\Omega)$.

Remark 3.2.2 The Euclidean case of this inequality was obtained by Kapitanski and Laptev [54]. In (3.18), $(xa_k^{-1})'_j = x_{j'} - a_{kj}'$ denotes the j^{th} component of xa_k^{-1}.

Proof of Theorem 3.2.1. Let us introduce a vector-field $\mathcal{A}(x) = (\mathcal{A}_1(x), \dots, \mathcal{A}_N(x))$ to be specified later. Also let λ be a real parameter for optimisation. We start with the inequality

$$0 \leq \int_\Omega \sum_{j=1}^N (|X_j u - \lambda \mathcal{A}_j u|^2) dx$$

$$= \int_\Omega \left(|\nabla_{\mathbb{G}} u|^2 - 2\lambda \mathrm{Re} \sum_{j=1}^N \overline{\mathcal{A}_j u} X_j u + \lambda^2 \sum_{j=1}^N |\mathcal{A}_j|^2 |u|^2\right) dx.$$

By using the integration by parts we get

$$-\int_{\Omega}\left(\lambda^2\sum_{j=1}^{N}|\mathcal{A}_j|^2+\lambda \mathrm{d}iv_{\mathbb{G}}\mathcal{A}\right)|u|^2dx\le\int_{\Omega}|\nabla_{\mathbb{G}}u|^2dx. \tag{3.19}$$

We differentiate the integral on the left-hand side with respect to λ for optimising it, yielding

$$2\lambda|\mathcal{A}|^2+\mathrm{d}iv_{\mathbb{G}}\mathcal{A}=0,$$

for all $x\in\Omega$. This is a restriction on $\mathcal{A}(x)$ giving $\frac{\mathrm{d}iv_{\mathbb{G}}\mathcal{A}(x)}{|\mathcal{A}(x)|^2}=const$. For $\lambda=\frac{1}{2}$ we get

$$\mathrm{d}iv_{\mathbb{G}}\mathcal{A}(x)=-|\mathcal{A}(x)|^2. \tag{3.20}$$

Then putting (3.20) in (3.19) we have the following Hardy inequality

$$\frac{1}{4}\int_{\Omega}\sum_{j=1}^{N}|\mathcal{A}_j(x)|^2|u|^2dx\le\int_{\Omega}|\nabla_{\mathbb{G}}u|^2dx. \tag{3.21}$$

Now if we assume that $\mathcal{A}=\nabla_{\mathbb{G}}\phi$ for some function ϕ, then (3.20) becomes

$$\mathcal{L}\phi+|\nabla_{\mathbb{G}}\phi|^2=0.$$

It follows that the function is harmonic (with respect to the sub-Laplacian).

$$w=e^{\phi}\ge 0.$$

Then w is a constant >0 or has a singularity. Let us consider

$$w:=\sum_{k=1}^{m}\frac{1}{|(xa_k^{-1})\prime|^{N-2}},$$

and then take

$$\phi(x)=\ln(w).$$

Therefore

$$\mathcal{A}(x)=\nabla_{\mathbb{G}}(\ln w)=\frac{1}{w}\nabla_{\mathbb{G}}(\sum_{k=1}^{m}|(xa_k^{-1})'|^{2-N})$$

$$=\frac{1}{w}\sum_{k=1}^{m}\nabla_{\mathbb{G}}\left(\sum_{j=1}^{N}((xa_k^{-1})_{j\prime})^2\right)^{\frac{2-N}{2}}$$

$$=-\frac{N-2}{w}\left(\sum_{k=1}^{m}\frac{(xa_k^{-1})\prime}{|(xa_k^{-1})\prime|^N}\right),$$

and

$$|\mathcal{A}(x)|^2 = \sum_{j=1}^{N} |\mathcal{A}_j(x)|^2 = \left(\frac{N-2}{w}\right)^2 \sum_{j=1}^{N} \left|\sum_{k=1}^{m} \frac{(xa_k^{-1})_{j\prime}}{|(xa_k^{-1})\prime|^N}\right|^2.$$

This completes the proof of Theorem 3.2.1.

We then also obtain the corresponding uncertainty principle.

Corollary 3.2.3 *Let $\Omega \subset \mathbb{G}$ be an open set, where $\mathbb{G}$ is a stratified group with N being the dimension of the first stratum. Let $N \geq 3$, $x = (x', x'') \in \mathbb{G}$ with $x' = (x'_1, \dots, x'_N)$ being in the first stratum of $\mathbb{G}$. Let $a_k \in \mathbb{G}, k = 1, \dots, m,$ be the singularities. Then we have*

$$\frac{N-2}{2} \int_\Omega |u|^2 dx \leq \left(\int_\Omega |\nabla_{\mathbb{G}} u|^2 dx\right)^{\frac{1}{2}} \left(\int_\Omega \frac{\left(\sum_{k=1}^{m} \frac{1}{|(xa_k^{-1})\prime|^{N-2}}\right)^2}{\sum_{j=1}^{N} \left|\sum_{k=1}^{m} \frac{(xa_k^{-1})_{j\prime}}{|(xa_k^{-1})\prime|^N}\right|^2} |u|^2 dx \right)^{\frac{1}{2}}, \quad (3.22)$$

for all $u \in C_0^\infty(\Omega)$ and $1 < p_i < N$ for $i = 1, \dots, N$.

Proof of Corollary 3.2.3. By (3.18) and the Cauchy-Schwarz inequality we get

$$\int_\Omega |\nabla_{\mathbb{G}} u|^2 dx \int_\Omega \frac{\left(\sum_{k=1}^{m} \frac{1}{|(xa_k^{-1})\prime|^{N-2}}\right)^2}{\sum_{j=1}^{N} \left|\sum_{k=1}^{m} \frac{(xa_k^{-1})_{j\prime}}{|(xa_k^{-1})\prime|^N}\right|^2} |u|^2 dx$$

$$\geq \left(\frac{N-2}{2}\right)^2 \int_\Omega \frac{\sum_{j=1}^{N} \left|\sum_{k=1}^{m} \frac{(xa_k^{-1})_{j\prime}}{|(xa_k^{-1})\prime|^N}\right|^2}{\left(\sum_{k=1}^{m} \frac{1}{|(xa_k^{-1})\prime|^{N-2}}\right)^2} |u|^2 dx \int_\Omega \frac{\left(\sum_{k=1}^{m} \frac{1}{|(xa_k^{-1})\prime|^{N-2}}\right)^2}{\sum_{j=1}^{N} \left|\sum_{k=1}^{m} \frac{(xa_k^{-1})_{j\prime}}{|(xa_k^{-1})\prime|^N}\right|^2} |u|^2 dx$$

$$\geq \left(\frac{N-2}{2}\right)^2 \left(\int_\Omega |u|^2 dx\right)^2.$$

The proof is complete.

3.3 Many-particle Hardy type inequality

In this section, we obtain the horizontal many-particle Hardy-type inequalities for $n \geq 1$ on the stratified groups. We consider that there are n particles, where n is a positive integer. Let $\mathbb{G}^n$ be the product

$$\mathbb{G}^n := \overbrace{\mathbb{G} \times \dots \times \mathbb{G}}^{n}.$$

We consider $x = (x_1, \dots, x_n) \in \mathbb{G}^n$, with $x_j \in \mathbb{G}$. Let $x \in \mathbb{G}^n$ with $x' = (x'_1, \dots, x'_n)$ and $x'_i = (x'_{i1}, \dots, x'_{iN})$ being the coordinates on the first stratum of $\mathbb{G}$ for $i = 1, \dots, n$. The distance between particles $x_i, x_j \in \mathbb{G}$ can be defined by

$$r_{ij} := |(x_i x_j^{-1})'| = |x'_i - x'_j| = \sqrt{\sum_{k=1}^{N} (x'_{ik} - x'_{jk})^2}.$$

We will use the following notation

$$\nabla_{\mathbb{G}_i} = (X_{i1}, \dots, X_{iN})$$

for the horizontal gradient associated to the i-th particle. We denote

$$\nabla_{\mathbb{G}^n} := (\nabla_{\mathbb{G}_1}, \dots, \nabla_{\mathbb{G}_n}),$$

and

$$\mathcal{L}_i = \sum_{k=1}^{N} X_{ik}^2$$

is the sub-Laplacian associated to the i-th particle. We note that

$$\mathcal{L} = \sum_{i=1}^{N} \mathcal{L}_i.$$

We recall a simple but crucial inequality on $\mathbb{R}^m$.

L e m m a 3.3.1 *Let* $m \geq 1$, *and let*

$$\mathcal{A} = (\mathcal{A}_1(x), \dots, \mathcal{A}_m(x))$$

be a mapping in $\mathcal{A}: \mathbb{R}^m \to \mathbb{R}^m$ *whose components and their first derivatives are uniformly bounded in* $\mathbb{R}^m$. *Then for* $u \in C_0^1(\mathbb{R}^m)$ *we have*

$$\int_{\mathbb{R}^m} |\nabla u|^2 dx \geq \frac{1}{4} \frac{\left(\int_{\mathbb{R}^m} div\mathcal{A}|u|^2 dx\right)^2}{\int_{\mathbb{R}^m} |\mathcal{A}|^2 |u|^2 dx}. \tag{3.23}$$

P r o o f o f L e m m a 3.3.1. We have

$$\left|\int_{\mathbb{R}^m} \mathrm{d}iv\mathcal{A}|u|^2 dx\right| = 2\left|\mathrm{Re}\int_{\mathbb{R}^m} \langle \mathcal{A}, \nabla u\rangle \overline{u} dx\right|$$

$$\leq 2\left(\int_{\mathbb{R}^m} |\mathcal{A}|^2 |u|^2 dx\right)^{1/2} \left(\int_{\mathbb{R}^m} |\nabla u|^2 dx\right)^{1/2}.$$

We have used the Cauchy-Schwarz inequality in the last line. The proof is finished by squaring this inequality.

T h e o r e m 3.3.2 *Let* $\Omega \subset \mathbb{G}^n$ *be an open set, where* $\mathbb{G}$ *is a stratified group with* N *being the dimension of the first stratum. Let* $N \geq 2$ *and* $n \geq 3$. *Let* $r_{ij} = |(x_i x_j^{-1})'| = |x'_i - x'_j|$. *Then we have*

$$\int_\Omega |\nabla_{\mathbb{G}^n} u|^2 dx \geq \frac{(N-2)^2}{n} \int_\Omega \sum_{1 \leq i < j \leq n} \frac{|u|^2}{r_{ij}^2} dx, \tag{3.24}$$

for all $u \in C^1(\Omega)$.

Remark 3.3.3 The Euclidean case of inequality (3.24) is obtained by M. Hoffmann-Ostenhof, T. Hoffmann-Ostenhof, A. Laptev, and J. Tidblom [55].

Proof of Theorem 3.3.2. Let us choose a mapping $\mathcal{B}_1$ in the following form

$$\mathcal{B}_1(x'_i, x'_j) := \frac{(x_i x_j^{-1})'}{r_{ij}^2}, \quad 1 \le i < j \le n.$$

And putting the mapping $\mathcal{B}_1$ in (3.23) we have

$$\int_\Omega |(\nabla_{\mathbb{G}_i} - \nabla_{\mathbb{G}_j})u|^2 dx \ge \frac{1}{4} \frac{\left(\int_\Omega \left((\mathrm{d}iv_{\mathbb{G}_i} - \mathrm{d}iv_{\mathbb{G}_j})\mathcal{B}_1\right)|u|^2 dx\right)^2}{\int_\Omega |\mathcal{B}_1|^2 |u|^2 dx}$$

$$= \frac{1}{4} \frac{\left(\int_\Omega \frac{2(N-2)}{|(x_i x_j^{-1})'|^2} |u|^2 dx\right)^2}{\int_\Omega \frac{|u|^2}{|(x_i x_j^{-1})'|^2} dx}$$

$$= (N-2)^2 \int_\Omega \frac{|u|^2}{r_{ij}^2} dx. \tag{3.25}$$

Also, we introduce another mapping $\mathcal{B}_2$

$$\mathcal{B}_2(x) := \frac{\sum_{j=1}^n x'_j}{\left|\sum_{j=1}^n x'_j\right|^2},$$

and

$$\mathrm{d}iv_{\mathbb{G}_i}\mathcal{B}_2 = \nabla_{\mathbb{G}_i} \cdot \mathcal{B}_2 = \sum_{k=1}^N X_{ik} \left(\frac{\sum_{j=1}^n x_{jk'}}{|\sum_{j=1}^n x_{j'}|^2}\right)$$

$$= \frac{Nn|\sum_{j=1}^n x_{j'}|^2 - 2n\left((\sum_{j=1}^n x_{j1'})^2 + \cdots + (\sum_{j=1}^n x_{jN'})^2\right)}{|\sum_{j=1}^n x_{j'}|^4}$$

$$= \frac{Nn - 2n}{|\sum_{j=1}^n x_{j'}|^2}.$$

As before we put the mapping $\mathcal{B}_2$ in (3.23) and using above computation yielding

$$\int_\Omega \left|\sum_{i=1}^n \nabla_{\mathbb{G}_i} u\right|^2 dx \ge \frac{1}{4} \frac{\left(\int_\Omega (\sum_{i=1}^n \mathrm{d}iv_{\mathbb{G}_i}\mathcal{B}_2)|u|^2 dx\right)^2}{\int_\Omega |\mathcal{B}_2|^2 |u|^2 dx}$$

$$= \frac{1}{4} \frac{\left(\int_\Omega \sum_{i=1}^n \frac{Nn-2n}{|\sum_{j=1}^n x_{j'}|^2} |u|^2 dx \right)^2}{\int_\Omega \frac{|u|^2}{\left|\sum_{j=1}^n x'_j\right|^2} dx}$$

$$= \frac{(N-2)^2 n^4}{4} \int_\Omega \frac{|u|^2}{\left|\sum_{j=1}^n x'_j\right|^2} dx. \tag{3.26}$$

Adding inequalities (3.25) and (3.26) and using the identity

$$n \sum_{i=1}^n |\nabla_{\mathbb{G}_i} u|^2 = \sum_{1 \le i < j \le n} \left| \nabla_{\mathbb{G}_i} u - \nabla_{\mathbb{G}_j} u \right|^2 + \left| \sum_{i=1}^n \nabla_{\mathbb{G}_i} u \right|^2,$$

we arrive at

$$\sum_{i=1}^n \int_\Omega |\nabla_{\mathbb{G}_i} u|^2 dx \ge \frac{(N-2)^2}{n} \int_\Omega \sum_{i<j} \frac{|u|^2}{r_{ij}^2} dx + \frac{(N-2)^2 n^3}{4} \int_\Omega \frac{|u|^2}{\left|\sum_{j=1}^n x'_j\right|^2} dx. \tag{3.27}$$

Because the last term on right-hand side is positive, we get

$$\sum_{i=1}^n \int_\Omega |\nabla_{\mathbb{G}_i} u|^2 dx \ge \frac{(N-2)^2}{n} \int_\Omega \sum_{i<j} \frac{|u|^2}{r_{ij}^2} dx.$$

Also we have

$$\begin{aligned} \sum_{i=1}^n |\nabla_{\mathbb{G}_i} u|^2 &= (\nabla_{\mathbb{G}_1} u)^2 + \cdots + (\nabla_{\mathbb{G}_n} u)^2 \\ &= |(\nabla_{\mathbb{G}_1} u, \dots, \nabla_{\mathbb{G}_n} u)|^2 \\ &= |\nabla_{\mathbb{G}^n} u|^2. \end{aligned}$$

The proof of Theorem 3.3.2 is complete.

The following theorem deals with the total separation of $n \ge 2$ particles.

Theorem 3.3.4 *Let $\Omega \subset \mathbb{G}^n$ be an open set, where $\mathbb{G}$ is a stratified group with N being the dimension of the first stratum. Let $\rho^2 := \sum_{i<j} |(x_i x_j^{-1})'|^2 = \sum_{i<j} |x_{i'} - x_{j'}|^2$ with $x'_i \ne x'_j$. Then we have*

$$\int_\Omega |\nabla_{\mathbb{G}} u|^2 dx = n \left(\frac{(n-1)}{2} N - 1 \right)^2 \int_\Omega \frac{|u|^2}{\rho^2} dx + \int_\Omega |\nabla_{\mathbb{G}} \rho^{-2\alpha} u|^2 \rho^{4\alpha} dx \tag{3.28}$$

for all $u \in C_0^\infty(\Omega)$ with $\alpha = \frac{2-(n-1)N}{4}$.

Remark 3.3.5. The Euclidean case of inequality (3.28) was obtained by Douglas Lundholm [56].

Proposition 3.3.6 *Let $\Omega \subset \mathbb{G}^n$ be an open set, where $\mathbb{G}$ is a stratified group with N being the dimension of the first stratum. Let $f: \Omega \to (0, \infty)$ be twice differentiable. Then for any function $u \in C_0^\infty(\Omega)$ and $\alpha \in \mathbb{R}$, we have*

$$\int_\Omega |\nabla_{\mathbb{G}} u|^2 dx = \int_\Omega \left(\alpha(1-\alpha)\frac{|\nabla_{\mathbb{G}} f|^2}{f^2} - \alpha\frac{\mathcal{L}f}{f}\right)|u|^2 dx + \int_\Omega |\nabla_{\mathbb{G}} v|^2 f^{2\alpha} dx, \tag{3.29}$$

where $v := f^{-\alpha}u$.

Proof of Proposition 3.3.6. Let us compute for $u = f^\alpha v$, that

$$\nabla_{\mathbb{G}} u = \alpha f^{\alpha-1}(\nabla_{\mathbb{G}} f)v + f^\alpha \nabla_{\mathbb{G}} v.$$

Then by squaring the above expression we have

$$|\nabla_{\mathbb{G}} u|^2 = \alpha^2 f^{2(\alpha-1)}|\nabla_{\mathbb{G}} f|^2|v|^2 + \mathrm{Re}(2\alpha v f^{2\alpha-1}(\nabla_{\mathbb{G}} f)\cdot(\nabla_{\mathbb{G}} v)) + f^{2\alpha}|\nabla_{\mathbb{G}} v|^2$$

$$= \alpha^2 f^{2(\alpha-1)}|\nabla_{\mathbb{G}} f|^2|v|^2 + \alpha f^{2\alpha-1}(\nabla_{\mathbb{G}} f)\cdot\nabla_{\mathbb{G}}|v|^2 + f^{2\alpha}|\nabla_{\mathbb{G}} v|^2.$$

By integrating this expression over Ω, we have

$$\int_\Omega |\nabla_{\mathbb{G}} u|^2 dx = \int_\Omega \alpha^2 f^{2(\alpha-1)}|\nabla_{\mathbb{G}} f|^2|v|^2 dx$$

$$+\int_\Omega \mathrm{Re}(\alpha f^{2\alpha-1}(\nabla_{\mathbb{G}} f)\cdot\nabla_{\mathbb{G}}|v|^2)dx + \int_\Omega f^{2\alpha}|\nabla_{\mathbb{G}} v|^2 dx$$

$$= \int_\Omega \alpha^2 f^{2(\alpha-1)}|\nabla_{\mathbb{G}} f|^2|v|^2 dx$$

$$-\alpha\int_\Omega \nabla_{\mathbb{G}}\cdot(f^{2\alpha-1}\nabla_{\mathbb{G}} f)|v|^2 dx + \int_\Omega f^{2\alpha}|\nabla_{\mathbb{G}} v|^2 dx.$$

We have used the integration by parts to the middle term on the right-hand side. Then

$$\nabla_{\mathbb{G}}\cdot(f^{2\alpha-1}\nabla_{\mathbb{G}} f) = (2\alpha-1)f^{2\alpha-2}|\nabla_{\mathbb{G}} f|^2 + f^{2\alpha-1}\mathcal{L}f,$$

and by using this fact we get

$$\int_\Omega |\nabla_{\mathbb{G}} u|^2 dx = \int_\Omega \alpha^2 f^{2(\alpha-1)}|\nabla_{\mathbb{G}} f|^2|v|^2 dx - \int_\Omega \alpha f^{2\alpha-1}\mathcal{L}f|v|^2 dx$$

$$-\int_\Omega \alpha(2\alpha-1)f^{2\alpha-2}|\nabla_{\mathbb{G}} f|^2|v|^2 dx + \int_\Omega f^{2\alpha}|\nabla_{\mathbb{G}} v|^2 dx.$$

Putting back $v = f^{-\alpha}u$ and collecting the terms we arrive at (3.29).

Proof of Theorem 3.3.4. The following computation gives

$$\nabla_{\mathbb{G}_k}\rho^2 = (X_{k1}\rho^2, \dots, X_{kN}\rho^2) = 2\sum_{k\neq j}^{n} (x_k x_j^{-1})',$$

where $\nabla_{\mathbb{G}_k} = (X_{k1}, \dots, X_{kN})$. Hence

$$\rho^2 = 2\sum_{k=1}^{n} \sum_{k\neq j}^{n} \nabla_{\mathbb{G}_k} \cdot (x_k x_j^{-1})' = 2n(n-1)N, \tag{3.30}$$

$$|\nabla_{\mathbb{G}}\rho^2|^2 = 8\sum_{1\le i<j\le n} \left|\left(x_k x_j^{-1}\right)'\right|^2 + 8\sum_{k=1}^{n} \sum_{1\le i<j\le n} (x_k x_i^{-1})' \cdot \left(x_k x_j^{-1}\right)' = 4n\rho^2, \tag{3.31}$$

where in the last step we used the identity

$$\sum_{k=1}^{n} \sum_{1\le i<j\le n} (x_k x_i^{-1})' \cdot (x_k x_j^{-1})' = \frac{n-2}{2}\sum_{1\le i<j\le n} |(x_i x_j^{-1})'|^2. \tag{3.32}$$

By putting (3.30) and (3.31) in Proposition 3.3.6 with $f = \rho^2$ we have

$$\int_\Omega |\nabla_{\mathbb{G}} u|^2 dx = 4n\alpha\left(\frac{2-(n-1)N}{2} - \alpha\right)\int_\Omega \frac{|u|^2}{\rho^2} dx + \int_\Omega |\nabla_{\mathbb{G}}\rho^{-2\alpha} u|^2 \rho^{4\alpha} dx.$$

To optimise we differentiate the integral

$$4n\alpha\left(\frac{2-(n-1)N}{2} - \alpha\right)\int_\Omega \frac{|u|^2}{\rho^2} dx$$

with respect to α, then we have

$$\frac{2-(n-1)N}{2} - 2\alpha = 0,$$

and

$$\alpha = \frac{2-(n-1)N}{4},$$

which completes the proof of Theorem 3.3.4.

3.4 Horizontal Hardy type inequalities with exponential weights

In this section, we get the horizontal Hardy inequality with exponential weights on $\mathbb{G}$.

Theorem 3.4.1 *Let $\Omega \subset \mathbb{G}$ be an open set, where $\mathbb{G}$ is a stratified group with $N \ge 3$ being the dimension of the first stratum. Let $x_0 \in \Omega$. Then we have*

$$\int_{\Omega} e^{-\frac{|(xx_0^{-1})'|^2}{4\lambda}}\left(\frac{(N-2)^2}{4|x'|^2}-\frac{N}{4\alpha}+\frac{|(xx_0^{-1})'|^2}{16\lambda^2}\right)|u|^2dx \leq \int_{\Omega} e^{-\frac{|(xx_0^{-1})'|^2}{4\lambda}}|\nabla_{\mathbb{G}}u|^2dx \qquad (3.33)$$

for all $u \in C^1(\Omega)$ *and for each* $\lambda > 0$.

Remark 3.4.2 Note that in the Euclidean case, this inequality is called two parabolic-type Hardy inequality, which was obtained by Zhang [57].

Proof of Theorem 3.4.1. Let us recall the horizontal Hardy inequality for all $v \in C^1(\Omega)$,

$$\frac{(N-2)^2}{4}\int_{\Omega}\frac{|v|^2}{|x'|^2}dx \leq \int_{\Omega}|\nabla_{\mathbb{G}}v|^2dx. \qquad (3.34)$$

Let $v = e^{-\frac{|(xx_0^{-1})'|^2}{8\lambda}}u$, then

$$\nabla_{\mathbb{G}}v = e^{-\frac{|(xx_0^{-1})'|^2}{8\lambda}}\nabla_{\mathbb{G}}u - \frac{(xx_0^{-1})'}{4\lambda}e^{-\frac{|(xx_0^{-1})'|^2}{8\lambda}}u,$$

for all $v \in C^1(\Omega)$. Then by inequality (3.34) we have

$$\frac{(N-2)^2}{4}\int_{\Omega} e^{-\frac{|(xx_0^{-1})'|^2}{4\lambda}}\frac{|u|^2}{|x'|^2}dx \leq \int_{\Omega} e^{-\frac{|(xx_0^{-1})'|^2}{4\lambda}}|\nabla_{\mathbb{G}}u|^2 + \frac{|(xx_0^{-1})'|^2}{16\lambda^2}e^{-\frac{|(xx_0^{-1})'|^2}{4\lambda}}|u|^2dx$$
$$-\mathrm{Re}\,\frac{1}{2\lambda}\int_{\Omega}(xx_0^{-1})'\cdot(\nabla_{\mathbb{G}}u)ue^{-\frac{|(xx_0^{-1})'|^2}{4\lambda}}dx. \qquad (3.35)$$

By the integration by parts in the last term of right-hand side of the inequality we have

$$\mathrm{Re}\int_{\Omega}(xx_0^{-1})'\cdot(\nabla_{\mathbb{G}}u)ue^{-\frac{|(xx_0^{-1})'|^2}{4\lambda}}dx = -\frac{1}{2}\int_{\Omega}\left(N-\frac{|(xx_0^{-1})'|^2}{2\lambda}\right)e^{-\frac{|(xx_0^{-1})'|^2}{4\lambda}}|u|^2dx. \qquad (3.36)$$

By putting equality (3.36) in (3.35) and rearranging it, we prove Theorem 3.4. 1.

3.5 Horizontal Hardy-Rellich type inequalities and embedding results

Theorem 3.5.1 *Let* $\mathbb{G}$ *be a homogeneous stratified group with* N *being the dimension of the first stratum, and let* $\alpha, \beta \in \mathbb{R}$. *Then for any* $f \in C_0^{\infty}(\mathbb{G}\backslash\{x' = 0\})$ *we have*

$$\left(\frac{N-(\alpha+\beta+3)}{2}\int_{\mathbb{G}}\frac{|\nabla_H f|^2}{|x'|^{\alpha+\beta+1}}dx+(\alpha+\beta+1)\int_{\mathbb{G}}\frac{(x'\cdot\nabla_H f)^2}{|x'|^{\alpha+\beta+3}}dx\right)^2$$
$$\leq\int_{\mathbb{G}}\frac{|\mathcal{L}f|^2}{|x'|^{2\beta}}dx\int_{\mathbb{G}}\frac{|\nabla_H f|^2}{|x'|^{2\alpha}}dx, \qquad (3.37)$$

where $|\cdot|$ *is the Euclidean norm on* $\mathbb{R}^N$*. Moreover, for* $\alpha+\beta+3\leq N$ *we have*

$$\frac{|N+\alpha+\beta-1|}{2}\int_{\mathbb{G}}\frac{|\nabla_H f|^2}{|x'|^{\alpha+\beta+1}}dx\leq\left(\int_{\mathbb{G}}\frac{|\mathcal{L}f|^2}{|x'|^{2\beta}}dx\right)^{\frac{1}{2}}\left(\int_{\mathbb{G}}\frac{|\nabla_H f|^2}{|x'|^{2\alpha}}dx\right)^{\frac{1}{2}}, \qquad (3.38)$$

with the sharp constant.

Let us define the following Sovolev type spaces on the stratified Lie group $\mathbb{G}$:

- Let $D^{1,2}_{\gamma}(\mathbb{G})$ be the completion of $C_0^\infty(\mathbb{G}\backslash\{x'=0\})$ with respect to the norm

$$\|f\|_{D^{1,2}_{\gamma}(\mathbb{G})}=\left(\int_{\mathbb{G}}\frac{|\nabla_H f|^2}{|x'|^{2\gamma}}dx\right)^{\frac{1}{2}}.$$

- Let $D^{2,2}_{\gamma}(\mathbb{G})$ be the completion of $C_0^\infty(\mathbb{G}\backslash\{x'=0\})$ with respect to the norm

$$\|f\|_{D^{2,2}_{\gamma}(\mathbb{G})}=\left(\int_{\mathbb{G}}\frac{|\mathcal{L}f|^2}{|x'|^{2\gamma}}dx\right)^{\frac{1}{2}}.$$

- Let $H^2_{\alpha,\beta}(\mathbb{G})$ be the completion of $C_0^\infty(\mathbb{G}\backslash\{x'=0\})$ with respect to the norm

$$\|f\|_{H^2_{\alpha,\beta}(\mathbb{G})}=\left(\int_{\mathbb{G}}\frac{|\nabla_H f|^2}{|x'|^{2\alpha}}+\frac{|\mathcal{L}f|^2}{|x'|^{2\beta}}dx\right)^{\frac{1}{2}}.$$

Theorem 3.5.2 *Let* $\mathbb{G}$ *be a homogeneous stratified group with* N *being the dimension of the first stratum, and let* $\alpha,\beta\in\mathbb{R}$*. We have the following continuous embedding*

$$H^2_{\alpha,\beta}(\mathbb{G})\subset D^{2,2}_{\frac{\alpha+\beta+1}{2}}(\mathbb{G}),$$

for $\alpha+\beta-1\neq N$.

$$D^{2,2}_{\alpha}(\mathbb{G})\subset D^{1,2}_{\alpha+1}(\mathbb{G}),$$

for $\alpha\leq\frac{N}{2}-2$ *with* $\alpha\neq\frac{N}{2}$.

In the abelian case $\mathbb{G}=(\mathbb{R}^n,+)$, we have $N=n$, $\nabla_H=\nabla=(\partial_{x_1},\ldots,\partial_{x_n})$, so (3.37) implies the Hardy-Rellich type inequality (see e.g. [58] and [59]) for $\mathbb{G}\equiv\mathbb{R}^n$:

$$\left(\frac{n-(\alpha+\beta+3)}{2}\int_{\mathbb{R}^n}\frac{|\nabla f|^2}{\|x\|^{\alpha+\beta+1}}dx+(\alpha+\beta+1)\int_{\mathbb{R}^n}\frac{(x\cdot\nabla f)^2}{\|x\|^{\alpha+\beta+3}}dx\right)^2$$
$$\leq\int_{\mathbb{R}^n}\frac{|\Delta f|^2}{\|x\|^{2\beta}}dx\int_{\mathbb{R}^n}\frac{|\nabla f|^2}{\|x\|^{2\alpha}}dx, \tag{3.39}$$

for all $f\in C_0^\infty(\mathbb{R}^n\backslash\{0\})$, and $\| x \|=\sqrt{x_1^2+\cdots+x_n^2}$.

When $\alpha=1$ and $\beta=0$, the inequality (3.38) gives the following stratified group version of Rellich's inequality

$$\int_{\mathbb{G}}\frac{|\nabla_H f|^2}{|x'|^2}dx\leq\left(\frac{2}{N}\right)^2\int_{\mathbb{G}}|\mathcal{L}f|^2dx,\quad 4\leq N, \tag{3.40}$$

with $\left(\frac{2}{N}\right)^2$ being the best constant.

Directly from the inequality (3.38), choosing α and β, we can obtain a number of Heisenberg-Pauli-Weyl type uncertainly inequalities which have various consequences and applications. For instance,

$$\frac{|N+2\alpha|}{2}\int_{\mathbb{G}}\frac{|\nabla_H f|^2}{|x'|^{2(\alpha+1)}}dx\leq\left(\int_{\mathbb{G}}\frac{|\mathcal{L}f|^2}{|x'|^{2(\alpha+1)}}dx\right)^{\frac{1}{2}}\left(\int_{\mathbb{G}}\frac{|\nabla_H f|^2}{|x'|^{2\alpha}}dx\right)^{\frac{1}{2}}$$

for $\alpha\leq\frac{N}{2}-2$ and any $f\in H^2_{\alpha,\alpha+1}(\mathbb{G})$.

$$\frac{|N-2|}{2}\int_{\mathbb{G}}|\nabla_H f|^2dx\leq\left(\int_{\mathbb{G}}|x'|^{2(\alpha+1)}|\nabla_H f|^2dx\right)^{\frac{1}{2}}\left(\int_{\mathbb{G}}\frac{|\mathcal{L}f|^2}{|x'|^{2\alpha}}dx\right)^{\frac{1}{2}}$$

for $3\leq N$ and any $f\in D_0^{1,2}(\mathbb{G})$.

$$\frac{N}{2}\int_{\mathbb{G}}\frac{|\nabla_H f|^2}{|x'|^2}dx\leq\left(\int_{\mathbb{G}}|\nabla_H f|^2dx\right)^{\frac{1}{2}}\left(\int_{\mathbb{G}}\frac{|\mathcal{L}f|^2}{|x'|^2}dx\right)^{\frac{1}{2}}$$

for any $f\in D_1^{1,2}(\mathbb{G})$.

$$\frac{N-1}{2}\int_{\mathbb{G}}\frac{|\nabla_H f|^2}{|x'|}dx\leq\left(\int_{\mathbb{G}}|x'|^2|\nabla_H f|^2dx\right)^{\frac{1}{2}}\left(\int_{\mathbb{G}}\frac{|\mathcal{L}f|^2}{|x'|^2}dx\right)^{\frac{1}{2}}$$

for $2\leq N$ and any $f\in D_{1/2}^{1,2}(\mathbb{G})$.

$$\frac{N-1}{2}\int_{\mathbb{G}}\frac{|\nabla_H f|^2}{|x'|}dx\leq\left(\int_{\mathbb{G}}|\nabla_H f|^2dx\right)^{\frac{1}{2}}\left(\int_{\mathbb{G}}|\mathcal{L}f|^2dx\right)^{\frac{1}{2}}$$

for $2\leq N$ and any $f\in D_{1/2}^{1,2}(\mathbb{G})$.

Proof of Theorem 3.5.1. For all $s \in \mathbb{R}^n$ we have

$$\int_{\mathbb{G}} \left|\frac{\nabla_H f}{|x'|^{\alpha}} + s\frac{x'}{|x'|^{\beta+1}}\mathcal{L}f\right|^2 dx \geq 0,$$

that is,

$$\int_{\mathbb{G}} \frac{|\nabla_H f|^2}{|x'|^{2\alpha}} dx + 2s\int_{\mathbb{G}} \frac{x'\cdot\nabla_H f}{|x'|^{\alpha+\beta+1}}\mathcal{L}f dx + s^2\int_{\mathbb{G}} \frac{|\mathcal{L}f|^2}{|x'|^{2\beta}} dx \geq 0. \tag{3.41}$$

Since

$$\int_{\mathbb{G}} \frac{x'\cdot\nabla_H f}{|x'|^{\alpha+\beta+1}}\mathcal{L}f dx = \int_{\mathbb{G}} \mathrm{d}iv_H(\nabla_H f)\left(\frac{x'\cdot\nabla_H f}{|x'|^{\alpha+\beta+1}}\right) dx$$

by using the divergence theorem we obtain

$$\int_{\mathbb{G}} \mathrm{d}iv_H(\nabla_H f)\left(\frac{x'\cdot\nabla_H f}{|x'|^{\alpha+\beta+1}}\right) dx = -\frac{1}{2}\int_{\mathbb{G}} \frac{x'}{|x'|^{\alpha+\beta+1}}\cdot\nabla_H(|\nabla_H f|^2)dx$$
$$-\int_{\mathbb{G}} \frac{|\nabla_H f|^2}{|x'|^{\alpha+\beta+1}} dx + (\alpha+\beta+1)\int_{\mathbb{G}} \frac{(x'\cdot\nabla_H f)^2}{|x'|^{\alpha+\beta+3}} dx.$$

Again by the divergence theorem we get

$$-\frac{1}{2}\int_{\mathbb{G}} \frac{x'}{|x'|^{\alpha+\beta+1}}\cdot\nabla_H(|\nabla_H f|^2)dx = \frac{N-(\alpha+\beta+1)}{2}\int_{\mathbb{G}} \frac{|\nabla_H f|^2}{|x'|^{\alpha+\beta+1}} dx.$$

Thus,

$$\int_{\mathbb{G}} \frac{x'\cdot\nabla_H f}{|x'|^{\alpha+\beta+1}}\mathcal{L}f dx = \frac{N-(\alpha+\beta+3)}{2}\int_{\mathbb{G}} \frac{|\nabla_H f|^2}{|x'|^{\alpha+\beta+1}} dx + (\alpha+\beta+1)\int_{\mathbb{G}} \frac{(x'\cdot\nabla_H f)^2}{|x'|^{\alpha+\beta+3}} dx. \tag{3.42}$$

Therefore, the equation (3.41) can be restated as

$$s^2\int_{\mathbb{G}} \frac{|\mathcal{L}f|^2}{|x'|^{2\beta}} dx + 2s\Big(\frac{N-(\alpha+\beta+3)}{2}\int_{\mathbb{G}} \frac{|\nabla_H f|^2}{|x'|^{\alpha+\beta+1}} dx$$
$$+(\alpha+\beta+1)\int_{\mathbb{G}} \frac{(x'\cdot\nabla_H f)^2}{|x'|^{\alpha+\beta+3}} dx\Big) + \int_{\mathbb{G}} \frac{|\nabla_H f|^2}{|x'|^{2\alpha}} dx \geq 0. \tag{3.43}$$

Denoting

$$a := \int_{\mathbb{G}} \frac{|\mathcal{L}f|^2}{|x'|^{2\beta}} dx,$$

$$b := \frac{N-(\alpha+\beta+3)}{2}\int_{\mathbb{G}} \frac{|\nabla_H f|^2}{|x'|^{\alpha+\beta+1}}dx + (\alpha+\beta+1)\int_{\mathbb{G}} \frac{(x'\cdot\nabla_H f)^2}{|x'|^{\alpha+\beta+3}}dx,$$

and

$$c := \int_{\mathbb{G}} \frac{|\nabla_H f|^2}{|x'|^{2\alpha}}dx,$$

we arrive at

$$as^2 + 2bs + c \geq 0, \tag{3.44}$$

which is equivalent to $b^2 - ac \leq 0$. Thus, we have

$$\left(\frac{N-(\alpha+\beta+3)}{2}\int_{\mathbb{G}} \frac{|\nabla_H f|^2}{|x'|^{\alpha+\beta+1}}dx + (\alpha+\beta+1)\int_{\mathbb{G}} \frac{(x'\cdot\nabla_H f)^2}{|x'|^{\alpha+\beta+3}}dx\right)^2 \leq \int_{\mathbb{G}} \frac{|\mathcal{L}f|^2}{|x'|^{2\beta}}dx \int_{\mathbb{G}} \frac{|\nabla_H f|^2}{|x'|^{2\alpha}}dx. \tag{3.45}$$

This shows the inequality (3.37). Now it remains to prove (3.38). It can be proved similarly. We refer [61] for a different proof of (3.38) .

Proof of Theorem 3.5.2. Since $N \neq \alpha+\beta-1$, from (3.38) we obtain

$$\int_{\mathbb{G}} \frac{|\nabla_H f|^2}{|x'|^{2\frac{(\alpha+\beta+1)}{2}}}dx \leq \frac{2}{|N+\alpha+\beta-1|}\left(\int_{\mathbb{G}} \frac{|\mathcal{L}f|^2}{|x'|^{2\beta}}dx\right)^{\frac{1}{2}}\left(\int_{\mathbb{G}} \frac{|\nabla_H f|^2}{|x'|^{2\alpha}}dx\right)^{\frac{1}{2}}$$
$$\leq \frac{2}{|N+\alpha+\beta-1|}\left(\int_{\mathbb{G}} \frac{|\mathcal{L}f|^2}{|x'|^{2\beta}}dx + \int_{\mathbb{G}} \frac{|\nabla_H f|^2}{|x'|^{2\alpha}}dx\right),$$

for all $f \in C_0^\infty(\mathbb{G}\backslash\{x'=0\})$. This proves Part (i). Part (ii) also implies from the inequality (3.38), namely assuming $\alpha+\beta+3 \leq N$ and letting $\beta = \alpha+1$, $\alpha \neq \frac{N}{2}$.

Corollary 3.5.3 *Let $\mathbb{G}$ be a homogeneous stratified group N being the dimension of the first stratum, and let $\alpha,\beta \in \mathbb{R}$. Then $f \in C_0^\infty(\mathbb{G}\backslash\{x'=0\})$, we have*

$$\frac{|N-\gamma|}{2}\left\|\frac{f}{|x'|^{\frac{\gamma}{2}}}\right\|^2_{L^2(\mathbb{G})} \leq \left\|\frac{\nabla_H f}{|x'|^{\alpha}}\right\|_{L^2(\mathbb{G})}\left\|\frac{f}{|x'|^{\beta}}\right\|_{L^2(\mathbb{G})} \tag{3.47}$$

where $\gamma = \alpha+\beta+1$ and then the constant $\frac{|N-\gamma|}{2}$ is sharp.

Proof of Corollary 3.5.3. Given $f \in C_0^\infty(\mathbb{G}\backslash x'=0)$ arbitrary and $\alpha,\beta \in \mathbb{R}$, we have

$$\int_{\mathbb{G}} \left|\frac{\nabla_H f}{|x'|^{\beta}} + s\frac{x'}{|x'|^{\alpha+1}}f\right|^2 dx \geqslant 0, \tag{3.48}$$

for every $s \in \mathbb{R}$.

$$\int_{\mathbb{G}} \frac{|\nabla_H f|^2}{|x'|^{2\beta}} dx + s^2 \int_{\mathbb{G}} \frac{|f|^2}{|x'|^{2\alpha}} dx + 2s \int_{\mathbb{G}} f \frac{x' \cdot \nabla_H f}{|x'|^{\gamma}} dx \geqslant 0 \tag{3.49}$$

by using divergence theorem

$$\int_{\mathbb{G}} f \frac{x' \cdot \nabla_H f}{|x'|^{\gamma}} dx = -\frac{N-\gamma}{2} \int_{\mathbb{G}} \frac{|f|^2}{|x'|^{\gamma}} dx,$$

$$a = \int_{\mathbb{G}} \frac{|f|^2}{|x'|^{2\alpha}} dx, \quad b = [N-\gamma] \int_{\mathbb{G}} \frac{|f|^2}{|x'|^{\gamma}}, \ c = \int_{\mathbb{G}} \frac{|\nabla_H f|^2}{|x'|^{2\beta}} dx,$$

$$as^2 - bs + c \geqslant 0,$$

This is equivalent to $b^2 - 4ac \leqslant 0$

$$[N-\gamma]^2 \left(\int_{\mathbb{G}} \frac{|f|^2}{|x'|^{\gamma}} \right)^2 \leqslant 4 \left(\int_{\mathbb{G}} \frac{|f|^2}{|x'|^{2\alpha}} dx \right) \left(\int_{\mathbb{G}} \frac{|\nabla_H f|^2}{|x'|^{2\beta}} dx \right). \tag{3.50}$$

Remark 3.5.4 Let us denote by $H^1_{\alpha,\beta}(\mathbb{G})$ the completion of $C_0^\infty(\mathbb{G} \backslash \{x' = 0\})$ with respect to the weighted Sobolev type norm

$$\|f\|_{H^1_{\alpha,\beta}} := \left(\int_{\mathbb{G}} \left[\frac{|f|^2}{|x'|^{2\alpha}} + \frac{|\nabla_H f|^2}{|x'|^{2\beta}} \right] dx \right)^{1/2}, \tag{3.51}$$

and by $L^2_\alpha(\mathbb{G})$ the completion of $C_0^\infty(\mathbb{G} \backslash \{x' = 0\})$ with respect to the weighted Lebesgue norm

$$\|f\|_{L^2_\alpha} := \left(\int_{\mathbb{G}} \frac{|f|^2}{|x'|^{2\alpha}} dx \right)^{1/2}. \tag{3.52}$$

Then the inequality (3.46) implies that, for $\gamma \neq N$, we have the continuous embedding

$$H^1_{\alpha,\beta}(\mathbb{G}) \subset L^2_{\gamma/2}(\mathbb{G}). \tag{3.53}$$

Moreover, since the right-hand side above is symmetric with respect to the parameters α, β we also have the continuous embedding

$$H^1_{\beta,\alpha}(\mathbb{G}) \subset L^2_{\gamma/2}(\mathbb{G}). \tag{3.54}$$

Corollary 3.5.5 *The inequalities below hold true with sharp constants:*

– *For any $f \in D^{1,2}(\mathbb{G})$ and $\alpha = 1, \beta = 0$ it follows that*

$$\left(\frac{N-2}{2}\right)^2 \int_{\mathbb{G}} \frac{|f|^2}{|x'|^2} dx \leq \int_{\mathbb{G}} |\nabla f|^2 dx. \tag{3.55}$$

- *For any* $f \in H^1_{\beta+1,\beta}(\mathbb{G})$ *and* $\alpha = \beta + 1$ *it follows that*

$$\left(\frac{N-2(\beta+1)}{2}\right)^2 \int_{\mathbb{G}} \frac{|f|^2}{|x'|^{2(\beta+1)}} dx \leq \int_{\mathbb{G}} \frac{|\nabla f|^2}{|x'|^{2\beta}} dx. \tag{3.56}$$

- *For any* $f \in H^1_{\alpha,\alpha+1}(\mathbb{G})$ *and* $\beta = \alpha + 1$ *it follows that*

$$\left(\frac{N-2(\alpha+1)}{2}\right)^2 \int_{\mathbb{G}} \frac{|f|^2}{|x'|^{2(\alpha+1)}} dx \leq \left(\int_{\mathbb{G}} \frac{|f|^2}{|x'|^{2\alpha}} dx\right)^{1/2} \left(\int_{\mathbb{G}} \frac{|\nabla f|^2}{|x'|^{2(\alpha+1)}} dx\right)^{1/2}. \tag{3.57}$$

- *For any* $f \in H^1_{-(\beta+1),\beta}(\mathbb{G})$ *and* $\alpha = -(\beta + 1)$, *then* $f \in L^2(\mathbb{G})$ *and*

$$\left(\frac{N}{2}\right) \int_{\mathbb{G}} |u|^2 dx \leq \left(\int_{\mathbb{G}} |x'|^{2(\beta+1)} |f|^2 dx\right)^{1/2} \left(\int_{\mathbb{G}} \frac{|\nabla f|^2}{|x'|^{2\beta}} dx\right)^{1/2}. \tag{3.58}$$

- *For any* $f \in H^1_{0,1}(\mathbb{G})$ *and* $\alpha = 0, \beta = 1$, *then* $f \in L^2_1(\mathbb{G})$ *and*

$$\left|\frac{N-2}{2}\right| \int_{\mathbb{G}} \frac{|u|^2}{|x'|^2} dx \leq \left(\int_{\mathbb{G}} |f|^2 dx\right)^{1/2} \left(\int_{\mathbb{G}} \frac{|\nabla f|^2}{|x'|^2} dx\right)^{1/2}. \tag{3.59}$$

- *For any* $f \in H^1_{-1,1}(\mathbb{G}), N > 1$ *and* $\alpha = -1, \beta = 1$, *then* $f \in L^2_{1/2}(\mathbb{G})$ *and*

$$\left(\frac{N-1}{2}\right) \int_{\mathbb{G}} \frac{|u|^2}{|x'|^2} dx \leq \left(\int_{\mathbb{G}} |x'|^2 |f|^2 dx\right)^{1/2} \left(\int_{\mathbb{G}} \frac{|\nabla f|^2}{|x'|^2} dx\right)^{1/2}. \tag{3.60}$$

- *For any* $f \in H^1(\mathbb{G}) = H^1_{0,0}(\mathbb{G}), N > 1$ *and* $\alpha = 0, \beta = 0$, *then* $f \in L^2_{1/2}(\mathbb{G})$ *and*

$$\left(\frac{N-1}{2}\right) \int_{\mathbb{G}} \frac{|u|^2}{|x'|^2} dx \leq \left(\int_{\mathbb{G}} |f|^2 dx\right)^{1/2} \left(\int_{\mathbb{G}} |\nabla f|^2 dx\right)^{1/2}. \tag{3.61}$$

4 HARDY TYPE INEQUALITIES AND SUB-LAPLACIAN FUNDAMENTAL SOLUTIONS

This chapter is devoted to present the Hardy and Rellich type inequalities on stratified groups with the $\mathcal{L}$-gauge weights. We recall that $\mathcal{L}$-gauge $d(x)$ is a homogeneous quasi-norm arising from the fundamental solution of the sub-Laplacian $\mathcal{L}$ such as

$$d(x) := \begin{cases} \varepsilon(x)^{\frac{1}{2-Q}}, & \text{for } x \neq 0, \\ 0, & \text{for } x \neq 0. \end{cases}$$

$d(x)^{2-Q}$ is a constant multiple of Folland's fundamental solution of the sub-Laplacian $\mathcal{L}$, with Q being the homogeneous dimension of the stratified group $\mathbb{G}$.

4.1 Weighted L^p-Hardy type inequalities with boundary terms

The main aim of this section is to give the generalised weighted L^p-Hardy type inequalities on stratified groups. We present a weighted L^p-Caffarelli-Kohn-Nirenberg type inequality with boundary term on the stratified group $\mathbb{G}$, which implies, in particular, the weighted L^p-Hardy type inequality. As consequences of those inequalities, we recover most of the known Hardy type inequalities and Heisenberg-Pauli-Weyl type uncertainty principles on the stratified group $\mathbb{G}$ [62].

Usually, unless we state explicitly otherwise, the functions u entering all the inequalities are complex-valued.

4.1.1 Weighted L^p-Caffarelli-Kohn-Nirenberg type inequality

We first present the following weighted L^p-Caffarelli-Kohn-Nirenberg type inequalities with boundary terms on the stratified Lie group $\mathbb{G}$ and then discuss their consequences. The proof of Theorem 4.1.1 is analogous to the proof of Davies and Hinz [8] but is now carried out in the case of the stratified Lie group $\mathbb{G}$. The boundary terms also give new addition to the Euclidean results. The classical Caffarelli-Kohn-Nirenberg inequalities in the Euclidean setting were obtained in [63].

Let $\mathbb{G}$ be a stratified group with N being the dimension of the first stratum, and let V be a real-valued function in $L^1_{loc}(\Omega)$ with partial derivatives of order up to 2 in $L^1_{loc}(\Omega)$, and such that $\mathcal{L}V$ is of one sign. Then we have:

Theorem 4.1.1 *Let Ω be an admissible domain in the stratified group $\mathbb{G}$ and let V be a real-valued function such that $\mathcal{L}V < 0$ holds a.e. in Ω. Then for any complex-valued $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$, and all $1 < p < \infty$, we have the inequality*

$$\left\| |\mathcal{L}V|^{\frac{1}{p}} u \right\|^p_{L^p(\Omega)} \leq p \left\| \frac{|\nabla_{\mathbb{G}} V|}{|\mathcal{L}V|^{\frac{p-1}{p}}} |\nabla_{\mathbb{G}} u| \right\|_{L^p(\Omega)} \left\| |\mathcal{L}V|^{\frac{1}{p}} u \right\|^{p-1}_{L^p(\Omega)} - \int_{\partial\Omega} |u|^p \langle \tilde{\nabla} V, dx \rangle. \quad (4.1)$$

Note that if u vanishes on the boundary $\partial\Omega$, then (4.1) extends the Davies and Hinz result to the weighted L^p-Hardy type inequality on stratified groups:

$$\left\| |\mathcal{L}V|^{\frac{1}{p}} u \right\|_{L^p(\Omega)} \leq p \left\| \frac{|\nabla_{\mathbb{G}} V|}{|\mathcal{L}V|^{\frac{p-1}{p}}} |\nabla_{\mathbb{G}} u| \right\|_{L^p(\Omega)}, \quad 1 < p < \infty. \tag{4.2}$$

P r o o f o f T h e o r e m 4.1.1. Let $v_\epsilon := (|u|^2 + \epsilon^2)^{\frac{1}{2}} - \epsilon$. Then $v_\epsilon^p \in C^2(\Omega) \cap C^1(\overline{\Omega})$ and using Green's first formula and the fact that $\mathcal{L}V < 0$ we get

$$\begin{aligned}
\int_\Omega |\mathcal{L}V| v_\epsilon^p dx &= -\int_\Omega \mathcal{L}V v_\epsilon^p dx \\
&= \int_\Omega (\widetilde{\nabla} V) v_\epsilon^p dx - \int_{\partial\Omega} v_\epsilon^p \langle \widetilde{\nabla} V, dx \rangle \\
&= \int_\Omega \nabla_{\mathbb{G}} V \cdot \nabla_{\mathbb{G}} v_\epsilon^p dx - \int_{\partial\Omega} v_\epsilon^p \langle \widetilde{\nabla} V, dx \rangle \\
&\leq \int_\Omega |\nabla_{\mathbb{G}} V| |\nabla_{\mathbb{G}} v_\epsilon^p| dx - \int_{\partial\Omega} v_\epsilon^p \langle \widetilde{\nabla} V, dx \rangle \\
&= p \int_\Omega \left(\frac{|\nabla_{\mathbb{G}} V|}{|\mathcal{L}V|^{\frac{p-1}{p}}} \right) |\mathcal{L}V|^{\frac{p-1}{p}} v_\epsilon^{p-1} |\nabla_{\mathbb{G}} v_\epsilon| dx - \int_{\partial\Omega} v_\epsilon^p \langle \widetilde{\nabla} V, dx \rangle,
\end{aligned}$$

where $(\widetilde{\nabla} u) v = \nabla_{\mathbb{G}} u \cdot \nabla_{\mathbb{G}} v$. We have

$$\nabla_{\mathbb{G}} v_\epsilon = (|u|^2 + \epsilon^2)^{-\frac{1}{2}} |u| \nabla_{\mathbb{G}} |u|,$$

since $0 \leq v_\epsilon \leq |u|$. Thus,

$$v_\epsilon^{p-1} |\nabla_{\mathbb{G}} v_\epsilon| \leq |u|^{p-1} |\nabla_{\mathbb{G}} |u||.$$

On the other hand, let $u(x) = R(x) + iI(x)$, where $R(x)$ and $I(x)$ denote the real and imaginary parts of u. We can restrict to the set where $u \neq 0$. Then we have

$$(\nabla_{\mathbb{G}} |u|)(x) = \frac{1}{|u|} (R(x) \nabla_{\mathbb{G}} R(x) + I(x) \nabla_{\mathbb{G}} I(x)) \quad \text{if} \quad u \neq 0. \tag{4.3}$$

Since

$$\left| \frac{1}{|u|} (R \nabla_{\mathbb{G}} R + I \nabla_{\mathbb{G}} I) \right|^2 \leq |\nabla_{\mathbb{G}} R|^2 + |\nabla_{\mathbb{G}} I|^2, \tag{4.4}$$

we get that $|\nabla_{\mathbb{G}} |u|| \leq |\nabla_{\mathbb{G}} u|$ a.e. in Ω. Therefore,

$$\int_{\Omega}|\mathcal{L}V|v_{\epsilon}^{p}dx \leq p\int_{\Omega}\left(\frac{|\nabla_{\mathbb{G}}V|}{|\mathcal{L}V|^{\frac{p-1}{p}}}|\nabla_{\mathbb{G}}u|\right)|\mathcal{L}V|^{\frac{p-1}{p}}|u|^{p-1}dx - \int_{\partial\Omega}v_{\epsilon}^{p}\langle\widetilde{\nabla}V, dx\rangle$$
$$\leq p\left(\int_{\Omega}\left(\frac{|\nabla_{\mathbb{G}}V|^{p}}{|\mathcal{L}V|^{(p-1)}}|\nabla_{\mathbb{G}}u|^{p}\right)dx\right)^{\frac{1}{p}}\left(\int_{\Omega}|\mathcal{L}V||u|^{p}dx\right)^{\frac{p-1}{p}} - \int_{\partial\Omega}v_{\epsilon}^{p}\langle\widetilde{\nabla}V, dx\rangle,$$

where we have used Hölder's inequality in the last line. Thus, when $\epsilon \to 0$, we obtain (4.1).

Here we have the horizontal $\boldsymbol{L^p}$-Caffarelli–Kohn–Nirenberg inequality with the boundary term:

Corollary 4.1.2 *Let Ω be an admissible domain in a stratified group $\mathbb{G}$ with $N \geq 3$ being the dimension of the first stratum, and let $\alpha, \beta \in \mathbb{R}$. Then for all $u \in C^2(\Omega\backslash\{x' = 0\}) \cap C^1(\overline{\Omega}\backslash\{x' = 0\})$, and any $1 < p < \infty$, we have*

$$\frac{|N-\gamma|}{p}\left\|\frac{u}{|x'|^{\frac{\gamma}{p}}}\right\|_{L^p(\Omega)}^{p} \leq \left\|\frac{\nabla_{\mathbb{G}}u}{|x'|^{\alpha}}\right\|_{L^p(\Omega)}\left\|\frac{u}{|x'|^{\frac{\beta}{p-1}}}\right\|_{L^p(\Omega)}^{p-1} - \frac{1}{p}\int_{\partial\Omega}|u|^p\langle\widetilde{\nabla}|x'|^{2-\gamma}, dx\rangle, \tag{4.5}$$

for $2 < \gamma < N$ with $\gamma = \alpha + \beta + 1$, and where $|\cdot|$ is the Euclidean norm on $\mathbb{R}^N$. In particular, if u vanishes on the boundary $\partial\Omega$, we have

$$\frac{|N-\gamma|}{p}\left\|\frac{u}{|x'|^{\frac{\gamma}{p}}}\right\|_{L^p(\Omega)}^{p} \leq \left\|\frac{\nabla_{\mathbb{G}}u}{|x'|^{\alpha}}\right\|_{L^p(\Omega)}\left\|\frac{u}{|x'|^{\frac{\beta}{p-1}}}\right\|_{L^p(\Omega)}^{p-1}. \tag{4.6}$$

Proof of Corollary 4.1.2. To obtain (4.5) from (4.1) , we take $V = |x'|^{2-\gamma}$. Then

$$|\nabla_{\mathbb{G}}V| = |2-\gamma||x'|^{1-\gamma}, \qquad |\mathcal{L}V| = |(2-\gamma)(N-\gamma)||x'|^{-\gamma},$$

and observe that $\mathcal{L}V = (2-\gamma)(N-\gamma)|x'|^{-\gamma} < 0$. To use (4.1) we calculate

$$\left\||\mathcal{L}V|^{\frac{1}{p}}u\right\|_{L^p(\Omega)}^{p} = |(2-\gamma)(N-\gamma)|\left\|\frac{u}{|x'|^{\frac{\gamma}{p}}}\right\|_{L^p(\Omega)}^{p},$$

$$\left\|\frac{|\nabla_{\mathbb{G}}V|}{|\mathcal{L}V|^{\frac{p-1}{p}}}\nabla_{\mathbb{G}}u\right\|_{L^p(\Omega)} = \frac{|2-\gamma|}{|(2-\gamma)(N-\gamma)|^{\frac{p-1}{p}}}\left\|\frac{|\nabla_{\mathbb{G}}u|}{|x'|^{\frac{\gamma-p}{p}}}\right\|_{L^p(\Omega)},$$

$$\left\||\mathcal{L}V|^{\frac{1}{p}}u\right\|_{L^p(\Omega)}^{p-1} = |(2-\gamma)(N-\gamma)|^{\frac{p-1}{p}}\left\|\frac{u}{|x'|^{\frac{\gamma}{p}}}\right\|_{L^p(\Omega)}^{p-1}.$$

Thus, (4.1) implies

$$\frac{|N-\gamma|}{p}\left\|\frac{u}{|x'|^{\frac{\gamma}{p}}}\right\|_{L^p(\Omega)}^p \leq \left\|\frac{\nabla_{\mathbb{G}} u}{|x'|^{\frac{\gamma-p}{p}}}\right\|_{L^p(\Omega)} \left\|\frac{u}{|x'|^{\frac{\gamma}{p}}}\right\|_{L^p(\Omega)}^{p-1} - \frac{1}{p}\int_{\partial\Omega} |u|^p \langle \widetilde{\nabla}|x'|^{2-\gamma}, dx\rangle.$$

If we denote $\alpha = \frac{\gamma-p}{p}$ and $\frac{\beta}{p-1} = \frac{\gamma}{p}$, we get (4.5).

4.1.2 Badiale-Tarantello conjecture

Theorem 4.1.1 also gives a new proof of the generalised Badiale-Tarantello conjecture [64] on the optimal constant in Hardy inequalities in $\mathbb{R}^n$ with weights taken with respect to a subspace.

Proposition 4.1.4 *Let $x = (x', x'') \in \mathbb{R}^N \times \mathbb{R}^{n-N}$, $1 \leq N \leq n$, $2 < \gamma < N$ and $\alpha, \beta \in \mathbb{R}$. Then for any $u \in C_0^\infty(\mathbb{R}^n \backslash \{x' = 0\})$ and all $1 < p < \infty$, we have*

$$\frac{|N-\gamma|}{p}\left\|\frac{u}{|x'|^{\frac{\gamma}{p}}}\right\|_{L^p(\mathbb{R}^n)}^p \leq \left\|\frac{\nabla u}{|x'|^{\alpha}}\right\|_{L^p(\mathbb{R}^n)} \left\|\frac{u}{|x'|^{\frac{\beta}{p-1}}}\right\|_{L^p(\mathbb{R}^n)}^{p-1}, \tag{4.7}$$

where $\gamma = \alpha + \beta + 1$ and $|x'|$ is the Euclidean norm $\mathbb{R}^N$. If $\gamma \neq N$ then the constant $\frac{|N-\gamma|}{p}$ is sharp.

The proof of Proposition 4.1.3 is similar to Corollary 4.1.2, so we sketch it only very briefly.

Proof of Proposition 4.1.4. Let us take $V = |x'|^{2-\gamma}$. We observe that $\Delta V = (2-\gamma)(N-\gamma)|x'|^{-\gamma} < 0$, as well as $|\nabla V| = |2-\gamma||x'|^{(1-\gamma)}$ and $|\Delta V| = |(2-\gamma)(N-\gamma)||x'|^{-\gamma}$. Then (4.1) with

$$\left\||\Delta V|^{\frac{1}{p}} u\right\|_{L^p(\mathbb{R}^n)}^p = |(2-\gamma)(N-\gamma)| \left\|\frac{u}{|x'|^{\frac{\gamma}{p}}}\right\|_{L^p(\mathbb{R}^n)}^p,$$

$$\left\|\frac{|\nabla V|}{|\Delta V|^{\frac{p-1}{p}}}\nabla u\right\|_{L^p(\mathbb{R}^n)} = \frac{|2-\gamma|}{|(2-\gamma)(N-\gamma)|^{\frac{p-1}{p}}}\left\|\frac{\nabla u}{|x'|^{\frac{\gamma-p}{p}}}\right\|_{L^p(\mathbb{R}^n)},$$

$$\left\||\Delta V|^{\frac{1}{p}} u\right\|_{L^p(\mathbb{R}^n)}^{p-1} = |(2-\gamma)(N-\gamma)|^{\frac{p-1}{p}} \left\|\frac{u}{|x'|^{\frac{\gamma}{p}}}\right\|_{L^p(\mathbb{R}^n)}^{p-1},$$

and denoting $\alpha = \frac{\gamma-p}{p}$ and $\frac{\beta}{p-1} = \frac{\gamma}{p}$, implies (4.7).

In particular, if we take $\beta = (\alpha+1)(p-1)$ and $\gamma = p(\alpha+1)$, then (4.7) implies

$$\frac{|N-p(\alpha+1)|}{p}\left\|\frac{u}{|x'|^{\alpha+1}}\right\|_{L^p(\mathbb{R}^n)} \leq \left\|\frac{\nabla u}{|x'|^{\alpha}}\right\|_{L^p(\mathbb{R}^n)}, \tag{4.8}$$

where $1 < p < \infty$, for all $u \in C_0^\infty(\mathbb{R}^n \backslash \{x' = 0\})$, $\alpha \in \mathbb{R}$, with sharp constant. When $\alpha = 0$, $1 < p < N$ and $2 \leq N \leq n$, the inequality (4.8) implies that

$$\left\|\frac{u}{|x'|}\right\|_{L^p(\mathbb{R}^n)} \leq \frac{p}{N-p}\|\nabla u\|_{L^p(\mathbb{R}^n)}, \tag{4.9}$$

which given another proof of the Badiale-Tarantello conjecture from Remark 2.3 [64].

As another consequence of Theorem 4.1.1 we obtain the local Hardy type inequality with the boundary term, with d being the $\mathcal{L}$-gauge.

Corollary 4.1.4. *Let $\Omega \subset \mathbb{G}$ with $0 \notin \partial\Omega$ be an admissible domain in a stratified group $\mathbb{G}$ of homogeneous dimension $Q \geq 3$. Let $0 > \alpha > 2 - Q$. Let $u \in C^1(\Omega \backslash \{0\}) \cap C(\overline{\Omega} \backslash \{0\})$. Then we have*

$$\frac{|Q+\alpha-2|}{p}\left\|d^{\frac{\alpha-2}{p}}|\nabla_{\mathbb{G}} d|^{\frac{2}{p}} u\right\|_{L^p(\Omega)} \leq \left\|d^{\frac{p+\alpha-2}{p}}|\nabla_{\mathbb{G}} d|^{\frac{2-p}{p}}|\nabla_{\mathbb{G}} u|\right\|_{L^p(\Omega)}$$
$$-\frac{1}{p}\left\|d^{\frac{\alpha-2}{p}}|\nabla_{\mathbb{G}} d|^{\frac{2}{p}} u\right\|_{L^p(\Omega)}^{1-p}\int_{\partial\Omega} d^{\alpha-1}|u|^p\langle\widetilde{\nabla} d, dx\rangle. \tag{4.10}$$

This extends the local Hardy type inequality that was obtained in [30, P. 518-520] for $p = 2$:

$$\frac{|Q+\alpha-2|}{2}\left\|d^{\frac{\alpha-2}{2}}|\nabla_{\mathbb{G}} d| u\right\|_{L^2(\Omega)} \leq \left\|d^{\frac{\alpha}{2}}|\nabla_{\mathbb{G}} u|\right\|_{L^2(\Omega)}$$
$$-\frac{1}{2}\left\|d^{\frac{\alpha-2}{2}}|\nabla_{\mathbb{G}} d| u\right\|_{L^2(\Omega)}^{-1}\int_{\partial\Omega} d^{\alpha-1}|u|^2\langle\widetilde{\nabla} d, dx\rangle. \tag{4.11}$$

Proof of Corollary 4.1.4. First, we can multiply both sides of the inequality (4.1) by $\left\||\mathcal{L}V|^{\frac{1}{p}} u\right\|_{L^p(\Omega)}^{1-p}$, so that we have

$$\left\||\mathcal{L}V|^{\frac{1}{p}} u\right\|_{L^p(\Omega)} \leq p\left\|\frac{|\nabla_{\mathbb{G}} V|}{|\mathcal{L}V|^{\frac{p-1}{p}}}|\nabla_{\mathbb{G}} u|\right\|_{L^p(\Omega)} -$$
$$\left\||\mathcal{L}V|^{\frac{1}{p}} u\right\|_{L^p(\Omega)}^{1-p}\int_{\partial\Omega}|u|^p\langle\widetilde{\nabla} V, dx\rangle. \tag{4.12}$$

Now, let us take $V = d^\alpha$. We have

$$\mathcal{L}d^{\alpha} = \nabla_{\mathbb{G}}(\nabla_{\mathbb{G}}\varepsilon^{\frac{\alpha}{2-Q}}) = \nabla_{\mathbb{G}}\left(\frac{\alpha}{2-Q}\varepsilon^{\frac{\alpha+Q-2}{2-Q}}\nabla_{\mathbb{G}}\varepsilon\right)$$
$$= \frac{\alpha(\alpha+Q-2)}{(2-Q)^2}\varepsilon^{\frac{\alpha-4+2Q}{2-Q}}|\nabla_{\mathbb{G}}\varepsilon|^2 + \frac{\alpha}{2-Q}\varepsilon^{\frac{\alpha+Q-2}{2-Q}}\mathcal{L}\varepsilon.$$

Since ε is the fundamental solution of $\mathcal{L}$, we have

$$\mathcal{L}d^{\alpha} = \frac{\alpha(\alpha+Q-2)}{(2-Q)^2}\varepsilon^{\frac{\alpha-4+2Q}{2-Q}}|\nabla_{\mathbb{G}}\varepsilon|^2 = \alpha(\alpha+Q-2)d^{\alpha-2}|\nabla_{\mathbb{G}}d|^2.$$

We can observe that $\mathcal{L}d^{\alpha} < 0$, and also the identities

$$\left\||\mathcal{L}d^{\alpha}|^{\frac{1}{p}}u\right\|_{L^p(\Omega)} = \alpha^{\frac{1}{p}}|Q+\alpha-2|^{\frac{1}{p}}\left\|d^{\frac{\alpha-2}{p}}|\nabla_{\mathbb{G}}d|^{\frac{2}{p}}u\right\|_{L^p(\Omega)},$$

$$\left\|\frac{|\nabla_{\mathbb{G}}d^{\alpha}|}{|\mathcal{L}d^{\alpha}|^{\frac{p-1}{p}}}|\nabla_{\mathbb{G}}u|\right\|_{L^p(\Omega)} = \alpha^{\frac{1}{p}}|Q+\alpha-2|^{\frac{1-p}{p}}\left\|d^{\frac{\alpha-2+p}{p}}|\nabla_{\mathbb{G}}d|^{\frac{2-p}{p}}|\nabla_{\mathbb{G}}u|\right\|_{L^p(\Omega)},$$

$$\left\||\mathcal{L}d^{\alpha}|^{\frac{1}{p}}u\right\|_{L^p(\Omega)}^{1-p}\int_{\partial\Omega}|u|^p\langle\widetilde{\nabla}d^{\alpha},dx\rangle = \alpha^{\frac{1}{p}}|Q+\alpha-2|^{\frac{1-p}{p}}\left\|d^{\frac{\alpha-2}{p}}|\nabla_{\mathbb{G}}d|^{\frac{2}{p}}u\right\|_{L^p(\Omega)}^{1-p}$$
$$\times\int_{\partial\Omega}d^{\alpha-1}|u|^p\langle\widetilde{\nabla}d,dx\rangle.$$

Using (4.12) we arrive at

$$\frac{|Q+\alpha-2|}{p}\left\|d^{\frac{\alpha-2}{p}}|\nabla_{\mathbb{G}}d|^{\frac{2}{p}}u\right\|_{L^p(\Omega)} \le \left\|d^{\frac{p+\alpha-2}{p}}|\nabla_{\mathbb{G}}d|^{\frac{2-p}{p}}|\nabla_{\mathbb{G}}u|\right\|_{L^p\Omega}$$
$$-\frac{1}{p}\left\|d^{\frac{\alpha-2}{p}}|\nabla_{\mathbb{G}}d|^{\frac{2}{p}}u\right\|_{L^p(\Omega)}^{1-p}\int_{\partial\Omega}d^{\alpha-1}|u|^p\langle\widetilde{\nabla}d,dx\rangle,$$

which implies (4.10).

The inequality (4.12) implies the following Heisenberg-Pauli-Weyl type uncertainty principle on stratified groups.

Corollary 4.1.5 *Let $\Omega\subset\mathbb{G}$ be admissible domain in a stratified group $\mathbb{G}$ and let $V\in C^2(\Omega)$ be real-valued. Then for any complex-valued function $u\in C^2(\Omega)\cap C^1(\overline{\Omega})$ we have*

$$\left\||\mathcal{L}V|^{-\frac{1}{p}}u\right\|_{L^p(\Omega)}\left\|\frac{|\nabla_{\mathbb{G}}V|}{|\mathcal{L}V|^{\frac{p-1}{p}}}|\nabla_{\mathbb{G}}u|\right\|_{L^p(\Omega)}$$
$$\ge\frac{1}{p}\|u\|_{L^p(\Omega)}^2+\frac{1}{p}\left\||\mathcal{L}V|^{-\frac{1}{p}}u\right\|_{L^p(\Omega)}\left\||\mathcal{L}V|^{\frac{1}{p}}u\right\|_{L^p(\Omega)}^{1-p}\int_{\partial\Omega}|u|^p\langle\widetilde{\nabla}V,dx\rangle. \quad (4.13)$$

In particular, if u vanishes on the boundary $\partial\Omega$, then we have

$$\left\| |\mathcal{L}V|^{-\frac{1}{p}} u \right\|_{L^p(\Omega)} \left\| \frac{|\nabla_{\mathbb{G}} V|}{|\mathcal{L}V|^{\frac{p-1}{p}}} |\nabla_{\mathbb{G}} u| \right\|_{L^p(\Omega)} \geq \frac{1}{p} \|u\|^2_{L^p(\Omega)}. \tag{4.14}$$

P r o o f o f C o r o l l a r y 4.1.5. By using the extended Hölder inequality and (4.12) we have

$$\begin{aligned}
&\left\| |\mathcal{L}V|^{-\frac{1}{p}} u \right\|_{L^p(\Omega)} \left\| \frac{|\nabla_{\mathbb{G}} V|}{|\mathcal{L}V|^{\frac{p-1}{p}}} |\nabla_{\mathbb{G}} u| \right\|_{L^p(\Omega)} \\
&\geq \frac{1}{p} \left\| |\mathcal{L}V|^{-\frac{1}{p}} u \right\|_{L^p(\Omega)} \left\| |\mathcal{L}V|^{\frac{1}{p}} u \right\|_{L^p(\Omega)} \\
&\qquad + \frac{1}{p} \left\| |\mathcal{L}V|^{-\frac{1}{p}} u \right\|_{L^p(\Omega)} \left\| |\mathcal{L}V|^{\frac{1}{p}} u \right\|^{1-p}_{L^p(\Omega)} \int_{\partial\Omega} |u|^p \langle \widetilde{\nabla} V, dx \rangle, \\
&\geq \frac{1}{p} \| |u|^2 \|_{L^{\frac{p}{2}}(\Omega)} + \frac{1}{p} \left\| |\mathcal{L}V|^{-\frac{1}{p}} u \right\|_{L^p(\Omega)} \left\| |\mathcal{L}V|^{\frac{1}{p}} u \right\|^{1-p}_{L^p(\Omega)} \int_{\partial\Omega} |u|^p \langle \widetilde{\nabla} V, dx \rangle. \\
&= \frac{1}{p} \|u\|^2_{L^p(\Omega)} + \frac{1}{p} \left\| |\mathcal{L}V|^{-\frac{1}{p}} u \right\|_{L^p(\Omega)} \left\| |\mathcal{L}V|^{\frac{1}{p}} u \right\|^{1-p}_{L^p(\Omega)} \int_{\partial\Omega} |u|^p \langle \widetilde{\nabla} V, dx \rangle,
\end{aligned}$$

proving (4.13).

By setting $V = |x'|^\alpha$ in the inequality (4.14), we recover the Heisenberg-Pauli-Weyl type uncertainty principle on stratified groups as in [65] and [31]:

$$\left(\int_\Omega |x'|^{2-\alpha} |u|^p dx \right) \left(\int_\Omega |x'|^{\alpha+p-2} |\nabla_{\mathbb{G}} u|^p dx \right) \geq \left(\frac{N+\alpha-2}{p} \right)^p \left(\int_\Omega |u|^p dx \right)^2.$$

In the abelian case $\mathbb{G} = (\mathbb{R}^n, +)$, taking $N = n \geq 3$, for $\alpha = 0$ and $p = 2$ this implies the classical Heisenberg-Pauli-Weyl uncertainty principle for all $u \in C_0^\infty(\mathbb{R}^n \backslash \{0\})$:

$$\left(\int_{\mathbb{R}^n} |x|^2 |u(x)|^2 dx \right) \left(\int_{\mathbb{R}^n} |\nabla u(x)|^2 dx \right) \geq \left(\frac{n-2}{2} \right)^2 \left(\int_{\mathbb{R}^n} |u(x)|^2 dx \right)^2.$$

By setting $V = d^\alpha$ in the inequality (4.14), we obtain another uncertainty type principle:

$$\left(\int_\Omega \frac{|u|^p}{d^{\alpha-2}|\nabla_{\mathbb{G}} d|^2}\,dx\right)\left(\int_\Omega d^{\alpha+p-2}|\nabla_{\mathbb{G}} d|^{2-p}|\nabla_{\mathbb{G}} u|^p\,dx\right) \geq \left(\frac{Q+\alpha-2}{p}\right)^p \left(\int_\Omega |u|^p\,dx\right)^2;$$

taking $p=2$ and $\alpha=0$ this yields

$$\left(\int_\Omega \frac{d^2}{|\nabla_{\mathbb{G}} d|^2}|u|^2\,dx\right)\left(\int_\Omega |\nabla_{\mathbb{G}} u|^2\,dx\right) \geq \left(\frac{Q-2}{2}\right)^2 \left(\int_\Omega |u|^2\,dx\right)^2.$$

4.2 Weighted L^p-Rellich type inequalities

In this section we establish weighted Rellich inequalities with boundary terms. We consider first the L^2 and then the L^p cases. The analogous L^2-Rellich inequality on $\mathbb{R}^n$ was proved by Schmincke [66] (and generalised by Bennett [67]).

Theorem 4.2.1 *Let Ω be an admissible domain in a stratified group $\mathbb{G}$ with $N \geq 2$ being the dimension of the first stratum. If a real-valued function $V \in C^2(\Omega)$ satisfies $\mathcal{L}V(x) < 0$ for all $x \in \Omega$, then for every $\epsilon > 0$ we have*

$$\left\|\frac{|V|}{|\mathcal{L}V|^{\frac{1}{2}}}\mathcal{L}u\right\|_{L^2(\Omega)}^2 \geq 2\epsilon\left\|V^{\frac{1}{2}}|\nabla_{\mathbb{G}} u|\right\|_{L^2(\Omega)}^2 + \epsilon(1-\epsilon)\left\||\mathcal{L}V|^{\frac{1}{2}}u\right\|_{L^2(\Omega)}^2 - \epsilon\int_{\partial\Omega}\left(|u|^2\langle\widetilde{\nabla}V,dx\rangle - V\langle\widetilde{\nabla}|u|^2,dx\rangle\right), \tag{4.15}$$

for all complex-valued functions $u \in C^2(\Omega)\cap C^1(\overline{\Omega})$. In particular, if u vanishes on the boundary $\partial\Omega$, we have

$$\left\|\frac{|V|}{|\mathcal{L}V|^{\frac{1}{2}}}\mathcal{L}u\right\|_{L^2(\Omega)}^2 \geq 2\epsilon\left\|V^{\frac{1}{2}}|\nabla_{\mathbb{G}} u|\right\|_{L^2(\Omega)}^2 + \epsilon(1-\epsilon)\left\||\mathcal{L}V|^{\frac{1}{2}}u\right\|_{L^2(\Omega)}^2.$$

Proof of Theorem 4.2.1. Using Green's second identity and that $\mathcal{L}V(x) < 0$ in Ω, we obtain

$$\int_\Omega |\mathcal{L}V||u|^2\,dx = -\int_\Omega V\mathcal{L}|u|^2\,dx - \int_{\partial\Omega}\left(|u|^2\langle\widetilde{\nabla}V,dx\rangle - V\langle\widetilde{\nabla}|u|^2,dx\rangle\right)$$
$$= -2\int_\Omega V\left(\mathrm{Re}(\overline{u}\mathcal{L}u) + |\nabla_{\mathbb{G}} u|^2\right)dx - \int_{\partial\Omega}\left(|u|^2\langle\widetilde{\nabla}V,dx\rangle - V\langle\widetilde{\nabla}|u|^2,dx\rangle\right).$$

Using the Cauchy-Schwartz inequality we get

$$\int_\Omega |\mathcal{L}V||u|^2\,dx \leq 2\left(\frac{1}{\epsilon}\int_\Omega \frac{|V|^2}{|\mathcal{L}V|}|\mathcal{L}u|^2\,dx\right)^{\frac{1}{2}}\left(\epsilon\int_\Omega |\mathcal{L}V||u|^2\,dx\right)^{\frac{1}{2}}$$

$$-2\int_\Omega V|\nabla_{\mathbb{G}}u|^2dx-\int_{\partial\Omega}(|u|^2\langle\widetilde{\nabla}V,dx\rangle-V\langle\widetilde{\nabla}|u|^2,dx\rangle)$$
$$\leq\frac{1}{\epsilon}\int_\Omega\frac{|V|^2}{|\mathcal{L}V|}|\mathcal{L}u|^2dx+\epsilon\int_\Omega|\mathcal{L}V||u|^2dx$$
$$-2\int_\Omega V|\nabla_{\mathbb{G}}u|^2dx-\int_{\partial\Omega}(|u|^2\langle\widetilde{\nabla}V,dx\rangle-V\langle\widetilde{\nabla}|u|^2,dx\rangle),$$

yielding (4.15).

Corollary 4.2.2 *Let $\mathbb{G}$ be a stratified group with N being the dimension of the first stratum. If $\alpha>-2$ and $N>\alpha+4$ then for all $u\in C_0^\infty(\mathbb{G}\backslash\{x'=0\})$ we have*

$$\int_{\mathbb{G}\backslash\{x'=0\}}\frac{|\mathcal{L}u|^2}{|x'|^\alpha}dx\geq\frac{(N+\alpha)^2(N-\alpha-4)^2}{16}\int_{\mathbb{G}\backslash\{x'=0\}}\frac{|u|^2}{|x'|^{\alpha+4}}dx. \tag{4.16}$$

Proof of Corollary 4.2.2. Let us take $V(x)=|x'|^{-(\alpha+2)}$ in Theorem 4.2.1, which can be applied since $x'=0$ is not in the support of u. Then we have

$$\nabla_{\mathbb{G}}V=-(\alpha+2)|x'|^{-\alpha-4}x',\qquad \mathcal{L}V=-(\alpha+2)(N-\alpha-4)|x'|^{-(\alpha+4)}.$$

Let us set $C_{N,\alpha}:=(\alpha+2)(N-\alpha-4)$. Observing that

$$\mathcal{L}V=-C_{N,\alpha}|x'|^{-(\alpha+4)}<0,$$

for $|x'|\neq0$, it follows from (4.2) that

$$\int_{\mathbb{G}\backslash\{x'=0\}}\frac{|\mathcal{L}u|^2}{|x'|^\alpha}dx\geq2C_{N,\alpha}\epsilon\int_{\mathbb{G}\backslash\{x'=0\}}\frac{|\nabla_{\mathbb{G}}u|^2}{|x'|^{\alpha+2}}dx$$
$$+C_{N,\alpha}^2\epsilon(1-\epsilon)\int_{\mathbb{G}\backslash\{x'=0\}}\frac{|u|^2}{|x'|^{\alpha+4}}dx. \tag{4.17}$$

To obtain (4.16), let us apply the L^p-Hardy type inequality (4.2) by taking $V(x)=|x'|^{\alpha+2}$ for $\alpha\in(-2,N-4)$, so that

$$\int_{\mathbb{G}\backslash\{x'=0\}}\frac{|\nabla_{\mathbb{G}}u|^2}{|x'|^{\alpha+2}}dx\geq\frac{(N-\alpha-4)^2}{4}\int_{\mathbb{G}\backslash\{x'=0\}}\frac{|u|^2}{|x'|^{\alpha+4}}dx,$$

and then choosing $\epsilon=(N+\alpha)/4(\alpha+2)$ for (4.3), which is the choice of ϵ that gives the maximum right-hand side.

We can now formulate the L^p-version of weighted L^p-Rellich type inequalities.

Theorem 4.2.3 *Let Ω be an admissible domain in a stratified group $\mathbb{G}$. If $0<V\in C(\Omega)$, $\mathcal{L}V<0$, and $\mathcal{L}(V^\sigma)\leq0$ on Ω for some $\sigma>1$, then for all $u\in C_0^\infty(\Omega)$ we have*

$$\left\| |\mathcal{L}V|^{\frac{1}{p}} u \right\|_{L^p(\Omega)} \leq \frac{p^2}{(p-1)\sigma+1} \left\| \frac{V}{|\mathcal{L}V|^{\frac{p-1}{p}}} \mathcal{L}u \right\|_{L^p(\Omega)}, \quad 1 \leq p < \infty. \tag{4.18}$$

Theorem 4.2.3 will follow by Lemma 4.2.5, by putting $C = \frac{(p-1)(\sigma-1)}{p}$ in Lemma 4.2.4.

Lemma 4.2.4 *Let Ω an admissible domain in a stratified group $\mathbb{G}$. If $V \geq 0$, $\mathcal{L}V < 0$, and there exists a constant $C \geq 0$ such that*

$$C \left\| |\mathcal{L}V|^{\frac{1}{p}} u \right\|_{L^p(\Omega)}^p \leq p(p-1) \left\| V^{\frac{1}{p}} |u|^{\frac{p-2}{p}} |\nabla_{\mathbb{G}} u|^{\frac{2}{p}} \right\|_{L^p(\Omega)}^p, \quad 1 < p < \infty, \tag{4.19}$$

for all $u \in C_0^\infty(\Omega)$, then we have

$$(1+C) \left\| |\mathcal{L}V|^{\frac{1}{p}} u \right\|_{L^p(\Omega)} \leq p \left\| \frac{V}{|\mathcal{L}V|^{\frac{p-1}{p}}} \mathcal{L}u \right\|_{L^p(\Omega)}, \tag{4.20}$$

for all $u \in C_0^\infty(\Omega)$. If $p = 1$ then the statement holds for $C = 0$.

Proof of Lemma 4.2.4. We can assume that u is real-valued by using the following identity:

$$\forall z \in \mathbb{C}: |z|^p = \left(\int_{-\pi}^{\pi} |\cos\vartheta|^p d\vartheta\right)^{-1} \int_{\pi}^{-\pi} |\mathrm{Re}(z)\cos\vartheta + \mathrm{I}m(z)\sin\vartheta|^p d\vartheta,$$

which can be proved by writing $z = r(\cos\phi + i\sin\phi)$ and simplifying.

Let $\epsilon > 0$ and set $u_\epsilon := (|u|^2 + \epsilon^2)^{p/2} - \epsilon^p$. Then $0 \leq u_\epsilon \in C_0^\infty$ and

$$\int_\Omega |\mathcal{L}V| u_\epsilon dx = -\int_\Omega (\mathcal{L}V) u_\epsilon dx = -\int_\Omega V \mathcal{L} u_\epsilon dx,$$

where

$$\begin{aligned} \mathcal{L}u_\epsilon &= \mathcal{L}\left((|u|^2+\epsilon^2)^{\frac{p}{2}} - \epsilon^p\right) = \nabla_{\mathbb{G}} \cdot (\nabla_{\mathbb{G}}((|u|^2+\epsilon^2)^{\frac{p}{2}} - \epsilon^p)) \\ &= \nabla_{\mathbb{G}}(p(|u|^2+\epsilon^2)^{\frac{p-2}{2}} u \nabla_{\mathbb{G}} u) \\ &= p(p-2)(|u|^2+\epsilon^2)^{\frac{p-4}{2}} u^2 |\nabla_{\mathbb{G}} u|^2 + p(|u|^2+\epsilon^2)^{\frac{p-2}{2}} |\nabla_{\mathbb{G}} u|^2 + p(|u|^2 \\ &\quad + \epsilon^2)^{\frac{p-2}{2}} u \mathcal{L}u. \end{aligned}$$

Then

$$\int_\Omega |\mathcal{L}V| u_\epsilon dx = -\int_\Omega \left(p(p-2)u^2(u^2+\epsilon^2)^{\frac{p-4}{2}} + p(u^2+\epsilon^2)^{\frac{p-2}{2}}\right) V |\nabla_{\mathbb{G}} u|^2 dx$$
$$-p\int_\Omega V u (u^2+\epsilon^2)^{\frac{p-2}{2}} \mathcal{L}u dx.$$

Hence

$$\int_\Omega |\mathcal{L}V|u_\epsilon + \left(p(p-2)u^2(u^2+\epsilon^2)^{\frac{p-4}{2}} + p(u^2+\epsilon^2)^{\frac{p-2}{2}}\right)V|\nabla_{\mathbb{G}}u|^2dx$$
$$\leq p\int_\Omega V|u|(u^2+\epsilon^2)^{\frac{p-2}{2}}|\mathcal{L}u|dx.$$

When $\epsilon \to 0$ the integrand on the right is bounded by $V(\max|u|^2 + 1)^{(p-1)/2}\max|\mathcal{L}u|$ and it is integrable because $u \in C_0^\infty(\Omega)$, and so the integral tends to $\int_\Omega V|u|^{p-1}|\mathcal{L}u|dx$ by the dominated convergence theorem. The integrand on the left is non-negative and tends to $|\mathcal{L}V||u|^p + p(p-1)V|u|^{p-2}|\nabla_{\mathbb{G}}u|^2$ pointwise, only for $u \neq 0$ when $p < 2$, otherwise for any x. It then follows by Fatou's lemma that

$$\left\||\mathcal{L}V|^{\frac{1}{p}}u\right\|^p_{L^p(\Omega)} + p(p-1)\left\|V^{\frac{1}{p}}|u|^{\frac{p-2}{p}}|\nabla_{\mathbb{G}}u|^{\frac{2}{p}}\right\|^p_{L^p(\Omega)} \leq p\left\|V^{\frac{1}{p}}|u|^{\frac{p-1}{p}}|\mathcal{L}u|^{\frac{1}{p}}\right\|^p_{L^p(\Omega)}.$$

By using (4.19), followed by the Hölder inequality, we obtain

$$(1+C)\left\||\mathcal{L}V|^{\frac{1}{p}}u\right\|^p_{L^p(\Omega)} \leq p\left\||\mathcal{L}V|^{(p-1)}V^{\frac{1}{p}}|u|^{\frac{p-1}{p}}|\mathcal{L}V|^{-(p-1)}|\mathcal{L}u|^{\frac{1}{p}}\right\|^p$$
$$\leq p\left\||\mathcal{L}V|^{\frac{1}{p}}u\right\|^{p-1}_{L^p(\Omega)}\left\|\frac{|V|}{|\mathcal{L}V|^{\frac{p-1}{p}}}\mathcal{L}u\right\|_{L^p(\Omega)}.$$

This implies (4.20).

Lemma 4.2.5 *Let Ω be an admissible domain in a stratified group $\mathbb{G}$. If $0 < V \in C(\Omega)$, $\mathcal{L}V < 0$, and $\mathcal{L}V^\sigma \leq 0$ on Ω for some $\sigma > 1$, then we have*

$$(\sigma-1)\int_\Omega |\mathcal{L}V||u|^p dx \leq p^2\int_{\{x\in\Omega, u(x)\neq 0\}} V|u|^{p-2}|\nabla_{\mathbb{G}}u|^2dx < \infty, \quad 1 < p < \infty, \tag{4.21}$$

for all $u \in C_0^\infty(\Omega)$.

Proof of Lemma 4.2.5. We shall use that

$$0 \geq \mathcal{L}(V^\sigma) = \sigma V^{\sigma-2}((\sigma-1)|\nabla_{\mathbb{G}}V|^2 + V\mathcal{L}V), \tag{4.22}$$

and hence

$$(\sigma-1)|\nabla_{\mathbb{G}}V|^2 \leq V|\mathcal{L}V|.$$

Now we use the inequality (4.2) for $p = 2$ to get

$$(\sigma-1)\int_\Omega |\mathcal{L}V||u|^2dx \leq 4(\sigma-1)\int_\Omega \frac{|\nabla_{\mathbb{G}}V|^2}{|\mathcal{L}V|}|\nabla_{\mathbb{G}}u|^2dx$$

$$\leq 4\int_{\Omega} V|\nabla_{\mathbb{G}}u|^2 dx = 4\int_{\{x\in\Omega; u(x)\neq 0, |\nabla_{\mathbb{G}}u|\neq 0\}} V|\nabla_{\mathbb{G}}u|^2 dx, \tag{4.23}$$

the last equality valid since $|\{x \in \Omega; u(x) = 0, |\nabla_{\mathbb{G}}u| \neq 0\}| = 0$. This proves Lemma 4.2.5 for $p = 2$.

For $p \neq 2$, put $v_\epsilon = (u^2 + \epsilon^2)^{p/4} - \epsilon^{p/2}$, and let $\epsilon \to 0$. Since $0 \leq v_\epsilon \leq |u|^{\frac{p}{2}}$, the left-hand side of (23), with u replaced by v_ϵ, tends to $(\sigma - 1)\int_{\Omega} |\mathcal{L}V||u|^p dx$ by the dominated convergence theorem. If $u \neq 0$, then

$$|\nabla_{\mathbb{G}} v_\epsilon|^2 V = \left|\frac{p}{2} u(u^2+\epsilon^2)^{\frac{p-4}{4}} \nabla_{\mathbb{G}} u\right|^2 V.$$

For $\epsilon \to 0$ we obtain

$$|\nabla_{\mathbb{G}} u|^p V = \frac{p^2}{4}|u|^{p-2}|\nabla_{\mathbb{G}} u|^2 V.$$

It follows as in the proof of Lemma 4.2.4, by using Fatou's lemma, that the right-hand side of (4.23) tends to

$$p^2\int_{\{x\in\Omega; u(x)\neq 0, |\nabla_{\mathbb{G}}u|\neq 0\}} V|u|^{p-2}|\nabla_{\mathbb{G}}u|^2 dx,$$

and this completes the proof.

Corollary 4.2.6 *Let $\mathbb{G}$ be a stratified group with N being the dimension of the first stratum. Then for any $2 < \alpha < N$ and all $u \in C_0^\infty(\mathbb{G}\backslash\{x' = 0\})$ we have the inequality*

$$\int_{\mathbb{G}} \frac{|u|^p}{|x'|^\alpha} dx \leq C^p_{(N,p,\alpha)} \int_{\mathbb{G}} \frac{|\mathcal{L}u|^p}{|x'|^{\alpha-2p}} dx, \tag{4.24}$$

where

$$C_{(N,p,\alpha)} = \frac{p^2}{(N-\alpha)((p-1)N+\alpha-2p)}. \tag{4.25}$$

Proof of Corollary 4.2.6. Let us choose $V = |x'|^{-(\alpha-2)}$ in Theorem 4.2.3, so that

$$\mathcal{L}V = -(\alpha-2)(N-\alpha)|x'|^{-\alpha},$$

and we note that when $2 < \alpha < N$, we have $\mathcal{L}V < 0$ for $|x'| \neq 0$. Now it follows from (4.18) that

$$(\alpha-2)^p(N-\alpha)^p \int_{\mathbb{G}} \frac{|u|^p}{|x'|^\alpha} dx \leq \frac{p^{2p}}{[(p-1)\sigma+1]^p} \int_{\mathbb{G}} \frac{|\mathcal{L}u|^p}{|x'|^{\alpha-2p}} dx. \tag{4.26}$$

By taking $\sigma = (N-2)/(\alpha-2)$, we arrive at

$$\int_{\mathbb{G}} \frac{|u|^p}{|x\prime|^\alpha} dx \leq \frac{p^{2p}}{(N-\alpha)^p((p-1)N+\alpha-2p)^p} \int_{\mathbb{G}} \frac{|\mathcal{L}u|^p}{|x\prime|^{\alpha-2p}} dx,$$

which proves (4.24)–(4.25).

Corollary 4.2.7 *Let $\mathbb{G}$ be a stratified Lie group and let $d = \varepsilon^{\frac{1}{2-Q}}$, where ε is the fundamental solution of the sub-Laplacian $\mathcal{L}$. Assume that $Q \geq 3$, $\alpha < 2$, and $Q + \alpha - 4 > 0$. Then for all $u \in C_0^\infty(\mathbb{G}\backslash\{0\})$ we have*

$$\frac{(Q+\alpha-4)^2(Q-\alpha)^2}{16} \int_{\mathbb{G}} d^{\alpha-4}|\nabla_{\mathbb{G}} d|^2 |u|^2 dx \leq \int_{\mathbb{G}} \frac{d^\alpha}{|\nabla_{\mathbb{G}} d|^2} |\mathcal{L}u|^2 dx. \tag{4.27}$$

The inequality (4.27) was obtained by Kombe [68], but now we get it as an immediate consequence of Theorem 4.2.3.

Proof of Corollary 4.2.7. Let us choose $V = d^{\alpha-2}$ in Theorem 4.2.3. Then

$$\mathcal{L}V = (\alpha-2)(Q+\alpha-4)d^{\alpha-4}|\nabla_{\mathbb{G}} d|^2.$$

Note that for $Q + \alpha - 4 > 0$ and $\alpha < 2$, we have $\mathcal{L}V < 0$ for all $x \neq 0$. If $p = 2$ then from (18) it follows that

$$(\alpha-2)^2(Q+\alpha-4)^2 \int_{\mathbb{G}} d^{\alpha-4}|\nabla_{\mathbb{G}} d|^2|u|^2 dx \leq \frac{16}{(\sigma+1)^2} \int_{\mathbb{G}} \frac{d^\alpha}{|\nabla_{\mathbb{G}} d|^2} |\mathcal{L}u|^2 dx.$$

By taking $\sigma = (Q - 2\alpha + 2)/(\alpha - 2)$ we get

$$\frac{(Q+\alpha-4)^2(Q-\alpha)^2}{16} \int_{\mathbb{G}} d^{\alpha-4}|\nabla_{\mathbb{G}} d|^2 |u|^2 dx \leq \int_{\mathbb{G}} \frac{d^\alpha}{|\nabla_{\mathbb{G}} d|^2} |\mathcal{L}u|^2 dx,$$

proving inequality (4.27).

Remark 4.2.8 In the abelian case, when $\mathbb{G} \equiv (\mathbb{R}^n, +)$ with $d = |x|$ being the Euclidean norm, and $\alpha = 0$ in inequality (4.27), we recover the classical Rellich inequality [69].

4.3 Hardy type inequalities on $\mathbb{H}_q$

In this section, we present a Hardy type inequality on the quaternion Heisenberg group. The proof of Theorem 3 relies on properties of the fundamental solution of the sub-Laplacian $\mathcal{L}$ on the quaternion Heisenberg group.

We recall the sub-Laplacian on $\mathbb{H}_q$

$$\mathcal{L} = \sum_{j=0}^3 X_j^2 = \Delta_x - 4|x|^2\Delta_t - 4\sum_{k=1}^3 (i_k x \cdot \nabla_x)\frac{\partial}{\partial t_k}$$

where

$$\Delta_x = \sum_{k=0}^{3} \frac{\partial^2}{\partial x_k^2}, \quad \text{and} \quad \Delta_t = \sum_{k=1}^{3} \frac{\partial^2}{\partial t_k^2}.$$

Note that the fundamental solution of the sub-Laplacian $\mathcal{L}$ on $\mathbb{H}_q$ was found by Tie and Wong. We restate their results in the following theorem.

Theorem 4.3.1 *The fundamental solution $\Gamma(\xi)$ of the sub-Laplacian $\mathcal{L}$ on the quaternion Heisenberg group $\mathbb{H}_q$ is given by*

$$\Gamma(\xi) := \Gamma(|x|, t) = \frac{2}{(2\pi)^{7/2}|x|^2} \int_{S^2} \frac{1}{(|x|^2 - i(t\cdot n))^3} d\sigma \tag{4.28}$$

where $\xi = (x,t) \in \mathbb{H}_q$, $n = (n_1, n_2, n_3)$ is a point on the unite sphere S^2 in $\mathbb{R}^3$ with center at the origin, and $d\sigma$ is the surface measure on S^2. That is,

$$\mathcal{L}\Gamma_\zeta = -\delta_\zeta, \tag{4.29}$$

where $\Gamma_\zeta(\xi) = \Gamma(\zeta^{-1} \circ \xi)$ and δ_ζ is the Dirac distribution at $\zeta \equiv (y, \tau) \in \mathbb{H}_q$.

The quaternion Heisenberg group is a special case of the model step two nilpotent Lie group. Moreover, it is a homogeneous Lie group with respect to the dilation

$$\delta_\lambda : \mathbb{R}^7 \to \mathbb{R}^7, \quad \delta_\lambda = (\lambda x, \lambda^2 t).$$

Thus,

$$d(\xi) = \frac{1}{\Gamma^{1/8}(\xi)}, \quad \xi = (x,t) \in \mathbb{H}_q, \tag{4.30}$$

is a homogeneous quasi-norm on $\mathbb{H}_q$ with respect to the dilation δ_λ [70].

Theorem 4.3.2 *Let $\alpha \in \mathbb{R}$, $\alpha > 2 - \beta$, $\beta > 2$. Then the following version of the Hardy inequality is valid:*

$$\left\| \Gamma^{\frac{\alpha}{2(2-\beta)}} |\nabla u| \right\|_{L_2(\mathbb{H}_q)} \geq \frac{|\beta + \alpha - 2|}{2} \left\| \Gamma^{\frac{\alpha-2}{2(2-\beta)}} |\nabla \Gamma^{\frac{1}{2-\beta}}| |u| \right\|_{L_2(\mathbb{H}_q)} \tag{4.31}$$

for any $u \in C_0^\infty(\mathbb{H}_q)$, where $\nabla = (X_0, X_1, X_2, X_3)$.

Proof of Theorem 4.3.2. Let $(\widetilde{\nabla} f)g := \sum_{k=0}^{3} X_k f X_k g$ for any differentiable functions f and g. Setting $u = d^\gamma q$ for some (real-valued) functions $d > 0$, q, and a constant $\gamma \neq 0$ to be chosen later, we have

$$(\widetilde{\nabla} u)u = (\widetilde{\nabla} d^\gamma q) d^\gamma q = \sum_{k=0}^{3} X_k (d^\gamma q) X_k (d^\gamma q)$$

$$= \gamma^2 d^{2\gamma - 2} \sum_{k=0}^{3} (X_k d)^2 q^2 + 2\gamma d^{2\gamma-1} q \sum_{k=0}^{3} X_k d X_k q + d^{2\gamma} \sum_{k=0}^{3} (X_k q)^2$$

$$= \gamma^2 d^{2\gamma-2}((\widetilde{\nabla} d)d)q^2 + 2\gamma d^{2\gamma-1} q(\widetilde{\nabla} d)q + d^{2\gamma}(\widetilde{\nabla} q)q.$$

Integrating by parts we observe that

$$2\gamma \int_{\mathbb{H}_q} d^{\alpha+2\gamma-1} q(\widetilde{\nabla} d) q dx = \frac{\gamma}{\alpha+2\gamma}\int_{\mathbb{H}_q} (\widetilde{\nabla} d^{\alpha+2\gamma})q^2 dx$$
$$= \frac{\gamma}{\alpha+2\gamma}\int_{\mathbb{H}_q} (\widetilde{\nabla} q^2) d^{\alpha+2\gamma} dx = -\frac{\gamma}{\alpha+2\gamma}\int_{\mathbb{H}_q} q^2 \mathcal{L} d^{\alpha+2\gamma} dx,$$

where we note that later on we will choose γ so that $d^{\alpha+2\gamma} = \Gamma$. Consequently, we get

$$\int_{\mathbb{H}_q} d^{\alpha}(\widetilde{\nabla} u) u dx = \gamma^2 \int_{\mathbb{H}_q} d^{\alpha+2\gamma-2}((\widetilde{\nabla} d)d)\, q^2 dx + \frac{\gamma}{\alpha+2\gamma}\int_{\mathbb{H}_q} (\widetilde{\nabla} d^{\alpha+2\gamma}) q^2 dx$$
$$+ \int_{\mathbb{H}_q} d^{\alpha+2\gamma}(\widetilde{\nabla} q) q dx$$
$$= \gamma^2 \int_{\mathbb{H}_q} d^{\alpha+2\gamma-2}((\widetilde{\nabla} d)d)\, q^2 dx - \frac{\gamma}{\alpha+2\gamma}\int_{\mathbb{H}_q} q^2 \mathcal{L} d^{\alpha+2\gamma} dx$$
$$+ \int_{\mathbb{H}_q} d^{\alpha+2\gamma}(\widetilde{\nabla} q) q dx$$
$$\geq \gamma^2 \int_{\mathbb{H}_q} d^{\alpha+2\gamma-2}((\widetilde{\nabla} d)d)\, q^2 dx - \frac{\gamma}{\alpha+2\gamma}\int_{\mathbb{H}_q} q^2 \mathcal{L} d^{\alpha+2\gamma} dx, \qquad (4.32)$$

since $d > 0$ and $(\widetilde{\nabla} q)q = |\nabla q|^2 \geq 0$. On the other hand, it can be readily checked that for a vector field X we have

$$\frac{\gamma}{\alpha+2\gamma} X^2(d^{\alpha+2\gamma}) = \gamma X(d^{\alpha+2\gamma-1} X d) = \frac{\gamma}{2-\beta} X(d^{\alpha+2\gamma+\beta-2} X(d^{2-\beta}))$$
$$= \frac{\gamma}{2-\beta}(\alpha+2\gamma+\beta-2) d^{\alpha+2\gamma+\beta-3}(Xd) X(d^{2-\beta})$$
$$+ \frac{\gamma}{2-\beta} d^{\alpha+2\gamma+\beta-2} X^2(d^{2-\beta})$$
$$= \gamma(\alpha+2\gamma+\beta-2) d^{\alpha+2\gamma-2}(Xd)^2$$
$$+ \frac{\gamma}{2-\beta} d^{\alpha+2\gamma+\beta-2} X^2(d^{2-\beta}).$$

Consequently, we get the equality

$$-\frac{\gamma}{\alpha+2\gamma}\mathcal{L} d^{\alpha+2\gamma} = -\gamma(\alpha+2\gamma+\beta-2) d^{\alpha+2\gamma-2}(\widetilde{\nabla} d) d -$$
$$\frac{\gamma}{2-\beta} d^{\alpha+2\gamma+\beta-2}\mathcal{L} d^{2-\beta}. \qquad (4.33)$$

Since $q^2 = d^{-2\gamma} u^2$, substituting (4.33) into (4.32) we obtain

$$\int_{\mathbb{H}_q} d^{\alpha}(\widetilde{\nabla} u) u dx \geq (-\gamma^2 - \gamma(\alpha+\beta-2)) \int_{\mathbb{H}_q} d^{\alpha-2}((\widetilde{\nabla} d) d) u^2 dx$$
$$-\frac{\gamma}{2-\beta}\int_{\mathbb{H}_q} (\mathcal{L} d^{2-\beta}) d^{\alpha+\beta-2} u^2 dx.$$

Taking $d = \Gamma^{\frac{1}{2-\beta}}$, $\beta > 2$, concerning the second term we observe that

$$\int_{\mathbb{H}_q} (\mathcal{L}\Gamma)\Gamma^{\frac{\alpha+\beta-2}{2-\beta}} u^2 dx = \left(\frac{1}{\Gamma(e)}\right)^{\frac{\alpha+\beta-2}{\beta-2}} u^2(e) = 0,\ \alpha > 2-\beta,\ \beta > 2, \quad (4.34)$$

since Γ is the fundamental solution of the sub-Laplacian $\mathcal{L}$. Here $e = (0,0,0,0)$ is the identity element of $\mathbb{H}_q$. Thus, with $d = \Gamma^{\frac{1}{2-\beta}}$, $\beta > 2$, we obtain

$$\int_{\mathbb{H}_q} \Gamma^{\frac{\alpha}{2-\beta}}(\widetilde{\nabla}u)u\, dx \geq (-\gamma^2 - \gamma(\alpha+\beta-2)) \int_{\mathbb{H}_q} \Gamma^{\frac{\alpha-2}{2-\beta}}(\widetilde{\nabla}\Gamma^{\frac{1}{2-\beta}})\Gamma^{\frac{1}{2-\beta}}u^2\, dx. \quad (4.35)$$

Now taking $\gamma = \frac{2-\beta-\alpha}{2}$, we arrive at (4.31).

Theorem 4.3.2 implies the following uncertainty principles:

Corollary 4.3.3 [Uncertainty principle on $\mathbb{H}_q$] *Let $\beta > 2$. Then for any $u \in C_0^\infty(\mathbb{H}_q)$ we have*

$$\int_{\mathbb{H}_q} \Gamma^{\frac{2}{2-\beta}}|\nabla\Gamma^{\frac{1}{2-\beta}}|^2|u|^2 dx \int_{\mathbb{H}_q} |\nabla u|^2 dx \geq \left(\frac{\beta-2}{2}\right)^2 \left(\int_{\mathbb{H}_q} |\nabla\Gamma^{\frac{1}{2-\beta}}|^2|u|^2 dx\right)^2, \quad (4.36)$$

and also

$$\int_{\mathbb{H}_q} \frac{\Gamma^{\frac{2}{2-\beta}}}{|\nabla\Gamma^{\frac{1}{2-\beta}}|^2}|u|^2 dx \int_{\mathbb{H}_q} |\nabla u|^2 dx \geq \left(\frac{\beta-2}{2}\right)^2 \left(\int_{\mathbb{H}_q} |u|^2 dx\right)^2. \quad (4.37)$$

Proof of Corollary 4.3.3. By taking $\alpha = 0$ in the inequality (31) we get

$$\int_{\mathbb{H}_q} \Gamma^{\frac{2}{2-\beta}}|\nabla\Gamma^{\frac{1}{2-\beta}}|^2|u|^2 dx \int_{\mathbb{H}_q} |\nabla u|^2 dx$$
$$\geq \left(\frac{\beta-2}{2}\right)^2 \int_{\mathbb{H}_q} \Gamma^{\frac{2}{2-\beta}}|\nabla\Gamma^{\frac{1}{2-\beta}}|^2|u|^2 dx \int_{\mathbb{H}_q} \frac{|\nabla\Gamma^{\frac{1}{2-\beta}}|^2}{\Gamma^{\frac{2}{2-\beta}}}|u|^2\, dx$$
$$\geq \left(\frac{\beta-2}{2}\right)^2 \left(\int_{\mathbb{H}_q} |\nabla\Gamma^{\frac{1}{2-\beta}}|^2|u|^2 dx\right)^2,$$

where we have used the Hölder inequality in the last line. This shows (4.36). The proof of (4.37) is similar.

4.4 Rellich type inequalities on $\mathbb{H}_q$

In this section, we present a version of the Rellich inequality on the quaternion Heisenberg group $\mathbb{H}_q$.

Theorem 4.4.1 *Let $\alpha \in \mathbb{R}$, $\beta > \alpha > 4-\beta$ and $\beta > 2$. Then the following version of the Rellich inequality is valid:*

$$\left\|\frac{\Gamma^{\frac{\alpha}{2(2-\beta)}}}{|\nabla\Gamma^{\frac{1}{2-\beta}}|}|\mathcal{L}u|\right\|_{L_2(\mathbb{H}_q)} \geq \frac{(\beta+\alpha-4)(\beta-\alpha)}{4}\left\|\Gamma^{\frac{\alpha-4}{2(2-\beta)}}|\nabla\Gamma^{\frac{1}{2-\beta}}|u\right\|_{L_2(\mathbb{H}_q)} \tag{4.38}$$

for any $u \in C_0^\infty(\mathbb{H}_q)$, where $\nabla = (X_0, X_1, X_2, X_3)$ is the gradient and $\mathcal{L}$ is the sub-Laplacian on the quaternion Heisenberg group $\mathbb{H}_q$ as defined in the introduction.

Proof of Theorem 4.4.1. A direct calculation shows that

$$\mathcal{L}\Gamma^{\frac{\alpha-2}{2-\beta}} = \sum_{k=0}^3 X_k^2\Gamma^{\frac{\alpha-2}{2-\beta}} = (\alpha-2)\sum_{k=0}^3 X_k\left(\Gamma^{\frac{\alpha-3}{2-\beta}}X_k\Gamma^{\frac{1}{2-\beta}}\right)$$

$$= (\alpha-2)(\alpha-3)\Gamma^{\frac{\alpha-4}{2-\beta}}\sum_{k=0}^3\left|X_k\Gamma^{\frac{1}{2-\beta}}\right|^2 + (\alpha-2)\Gamma^{\frac{\alpha-3}{2-\beta}}\sum_{k=0}^3 X_k\left(X_k\Gamma^{\frac{1}{2-\beta}}\right)$$

$$= (\alpha-2)(\alpha-3)\Gamma^{\frac{\alpha-4}{2-\beta}}\sum_{k=0}^3\left|X_k\Gamma^{\frac{1}{2-\beta}}\right|^2 + \frac{\alpha-2}{2-\beta}\Gamma^{\frac{\alpha-3}{2-\beta}}\sum_{k=0}^3 X_k\left(\Gamma^{\frac{\beta-1}{2-\beta}}X_k\Gamma\right)$$

$$= (\alpha-2)(\alpha-3)\Gamma^{\frac{\alpha-4}{2-\beta}}\sum_{k=0}^3\left|X_k\Gamma^{\frac{1}{2-\beta}}\right|^2$$

$$+\frac{(\alpha-2)(\beta-1)}{2-\beta}\Gamma^{\frac{\alpha-3}{2-\beta}}\Gamma^{-1}\sum_{k=0}^3 (X_k\Gamma^{\frac{1}{2-\beta}})(X_k\Gamma)$$

$$+\frac{\alpha-2}{2-\beta}\Gamma^{\frac{\beta+\alpha-4}{2-\beta}}\mathcal{L}\Gamma = (\alpha-2)(\alpha-3)\Gamma^{\frac{\alpha-4}{2-\beta}}\sum_{k=0}^3\left|X_k\Gamma^{\frac{1}{2-\beta}}\right|^2$$

$$+(\alpha-2)(\beta-1)\Gamma^{\frac{\alpha-4}{2-\beta}}\sum_{k=0}^3 (X_k\Gamma^{\frac{1}{2-\beta}})(X_k\Gamma^{\frac{1}{2-\beta}}) + \frac{\alpha-2}{2-\beta}\Gamma^{\frac{\beta+\alpha-4}{2-\beta}}\mathcal{L}\Gamma$$

$$= (\beta+\alpha-4)(\alpha-2)\Gamma^{\frac{\alpha-4}{2-\beta}}|\nabla\Gamma^{\frac{1}{2-\beta}}|^2 + \frac{\alpha-2}{2-\beta}\Gamma^{\frac{\beta+\alpha-4}{2-\beta}}\mathcal{L}\Gamma,$$

that is,

$$\mathcal{L}\Gamma^{\frac{\alpha-2}{2-\beta}} = (\beta+\alpha-4)(\alpha-2)\Gamma^{\frac{\alpha-4}{2-\beta}}|\nabla\Gamma^{\frac{1}{2-\beta}}|^2 + \frac{\alpha-2}{2-\beta}\Gamma^{\frac{\beta+\alpha-4}{2-\beta}}\mathcal{L}\Gamma. \tag{4.39}$$

As before we can assume that u is real-valued. Multiplying both sides of (4.39) by u^2 and integrating over $\mathbb{H}_q$, since Γ is the fundamental solution of $\mathcal{L}$ and $\beta+\alpha-4>0$, we get

$$\int_{\mathbb{H}_q} u^2\mathcal{L}\Gamma^{\frac{\alpha-2}{2-\beta}}\,dx = (\beta+\alpha-4)(\alpha-2)\int_{\mathbb{H}_q}\Gamma^{\frac{\alpha-4}{2-\beta}}|\nabla\Gamma^{\frac{1}{2-\beta}}|^2u^2\,dx. \tag{4.40}$$

On the other hand, integrating by parts, we have

$$\int_{\mathbb{H}_q} u^2 \mathcal{L}\Gamma^{\frac{\alpha-2}{2-\beta}}\, dx = \int_{\mathbb{H}_q} \Gamma^{\frac{\alpha-2}{2-\beta}} \mathcal{L}u^2\, dx = \int_{\mathbb{H}_q} \Gamma^{\frac{\alpha-2}{2-\beta}} (2u\mathcal{L}u + 2|\nabla u|^2)\, dx, \tag{4.41}$$

Combining (4.40) and (4.41) we obtain

$$\begin{aligned} &-2\int_{\mathbb{H}_q} \Gamma^{\frac{\alpha-2}{2-\beta}} u\mathcal{L}u dx + (\beta+\alpha-4)(\alpha-2)\int_{\mathbb{H}_q} \Gamma^{\frac{\alpha-4}{2-\beta}} |\nabla\Gamma^{\frac{1}{2-\beta}}|^2\, u^2 dx \\ &= 2\int_{\mathbb{H}_q} \Gamma^{\frac{\alpha-2}{2-\beta}} |\nabla u|^2 dx. \end{aligned} \tag{4.42}$$

By using (4.31) we establish

$$\begin{aligned} &-2\int_{\mathbb{H}_q} \Gamma^{\frac{\alpha-2}{2-\beta}} u\mathcal{L}u dx + (\beta+\alpha-4)(\alpha-2)\int_{\mathbb{H}_q} \Gamma^{\frac{\alpha-4}{2-\beta}} |\nabla\Gamma^{\frac{1}{2-\beta}}|^2\, |u|^2 dx \\ &\qquad \geq 2\left(\frac{\beta+\alpha-4}{2}\right)^2 \int_{\mathbb{H}_q} \Gamma^{\frac{\alpha-4}{2-\beta}} |\nabla\Gamma^{\frac{1}{2-\beta}}|^2 |u|^2\, dx. \end{aligned} \tag{4.43}$$

It follows that

$$-\int_{\mathbb{H}_q} \Gamma^{\frac{\alpha-2}{2-\beta}} u\mathcal{L}u dx \geq \left(\frac{\beta+\alpha-4}{2}\right)\left(\frac{\beta-\alpha}{2}\right) \int_{\mathbb{H}_q} \Gamma^{\frac{\alpha-4}{2-\beta}} |\nabla\Gamma^{\frac{1}{2-\beta}}|^2 |u|^2\, dx. \tag{4.44}$$

On the other hand, for any $\epsilon > 0$ Hölder's and Young's inequalities give

$$\begin{aligned} -\int_{\mathbb{H}_q} \Gamma^{\frac{\alpha-2}{2-\beta}} u\mathcal{L}u dx &\leq \left(\int_{\mathbb{H}_q} \Gamma^{\frac{\alpha-4}{2-\beta}} |\nabla\Gamma^{\frac{1}{2-\beta}}|^2 |u|^2 dx\right)^{\frac{1}{2}} \left(\int_{\mathbb{H}_q} \frac{\Gamma^{\frac{\alpha}{2-\beta}}}{|\nabla\Gamma^{\frac{1}{2-\beta}}|^2} |\mathcal{L}u|^2 dx\right)^{\frac{1}{2}} \\ &\leq \epsilon \int_{\mathbb{H}_q} \Gamma^{\frac{\alpha-4}{2-\beta}} |\nabla\Gamma^{\frac{1}{2-\beta}}|^2 |u|^2 dx + \frac{1}{4\epsilon}\int_{\mathbb{H}_q} \frac{\Gamma^{\frac{\alpha}{2-\beta}}}{|\nabla\Gamma^{\frac{1}{2-\beta}}|^2} |\mathcal{L}u|^2 dx. \end{aligned} \tag{4.45}$$

Inequalities (4.45) and (4.44) imply that

$$\int_{\mathbb{H}_q} \frac{\Gamma^{\frac{\alpha}{2-\beta}}}{|\nabla\Gamma^{\frac{1}{2-\beta}}|^2} |\mathcal{L}u|^2 dx \geq (-4\epsilon^2 + (\beta+\alpha-4)(\beta-\alpha)\epsilon) \int_{\mathbb{H}_q} \Gamma^{\frac{\alpha-4}{2-\beta}} |\nabla\Gamma^{\frac{1}{2-\beta}}|^2 |u|^2\, dx.$$

Taking $\epsilon = \frac{(\beta+\alpha-4)(\beta-\alpha)}{8}$, we arrive at

$$\int_{\mathbb{H}_q} \frac{\Gamma^{\frac{\alpha}{2-\beta}}}{|\nabla\Gamma^{\frac{1}{2-\beta}}|^2} |\mathcal{L}u|^2 dx \geq \frac{(\beta+\alpha-4)^2(\beta-\alpha)^2}{16} \int_{\mathbb{H}_q} \Gamma^{\frac{\alpha-4}{2-\beta}} |\nabla\Gamma^{\frac{1}{2-\beta}}|^2 |u|^2\, dx.$$

5 WEIGHTED ANISOTROPIC HARDY AND RELLICH TYPE INEQUALITIES FOR GENERAL VECTOR FIELDS

This chapter is devoted to the weighted anisotropic Hardy and Rellich type inequalities with boundary terms for general (real-valued) vector fields. The consequences recover many previously known results in different settings. The anisotropic Picone type identities play key roles in our proofs.

Recall the Hardy inequality for $\Omega \subset \mathbb{R}^n$ stating that

$$\int_{\Omega} |\nabla u|^p dx \geq C \int_{\Omega} \frac{|u|^p}{|x|^p} dx,\ u \in C_0^1(\Omega), \tag{5.1}$$

where ∇ is the Euclidean gradient and $p > 1$. It has been vastly studied by many authors and developed in different settings [71] and the references therein.

First, let us review some of the recent results:

- Hardy type inequalities in the setting of the *Heisenberg group* $\mathbb{H}^n$ have the following form

$$\int_{\mathbb{H}^n} |\nabla_H u|^2 dx \geq C \int_{\mathbb{H}^n} \frac{\psi_H^2}{\rho^2} |u|^2 dx,\ u \in C_0^1(\mathbb{H}^n \backslash \{0\}), \tag{5.2}$$

where ∇_H is a (horizontal) gradient associated to the sub-Laplacian, ψ_H and ρ are a weight function and a suitable distance from the origin, respectively. For example, Garofalo and Lanconelli in [18, P. 330-334], D'Ambrosio in [35, P. 513-514], Niu, Zhang and Wang in [72], and others have made a contribution to prove the above inequality and its extensions in $\mathbb{H}^n$.

- Hardy type inequalities in the setting of the *Carnot group* $\mathbb{G}$ can be given by the formula

$$\int_{\mathbb{G}} d^{\alpha} |\nabla_H u|^2 dx \geq C \int_{\mathbb{G}} d^{\alpha-2} |\nabla_H d|^2 |u|^2 dx,\ u \in C_0^{\infty}(\mathbb{G} \backslash \{0\}), \tag{5.3}$$

where ∇_H is the horizontal gradient on $\mathbb{G}$, $\alpha \in \mathbb{R}$, and d is a homogeneous norm associated with a fundamental solution for the sub-Laplacian. For instance, the Hardy type inequalities on $\mathbb{G}$ have been studied by Ruzhansky and Suragan in [61, P. 1815-1816], Kombe in [68, P. 255-256], Goldstein, Kombe and Yener in [71, P. 2015-2016], Wang and Niu in [73].

- Hardy type inequalities in the setting of *general vector fields* can be presented in the form

$$\int_{\Omega} |\nabla_X u|^p dx \geq C \int_{\Omega} \frac{|\nabla_X \phi|^p}{\phi^p} |u|^p dx,\ u \in C_0^1(\Omega), \tag{5.4}$$

where $\nabla_X := (X_1, \dots, X_N)$ and ϕ is any positive weight function. To the best of our knowledge, D'Ambrosio obtained first versions of Hardy type inequalities for general vector fields in [74].

Consider a family of real vector fields $\{X_k\}_{k=1}^N, N \leq n,$ on a smooth manifold M with dimension n and a volume form dx.

$$\int_\Omega W(x)|\nabla_X u|^p dx \geq \int_\Omega H(x)|u|^p dx, \ u \in C_0^1(\Omega),$$

with the hypothesis

$$-\nabla_X \cdot (W(x)|\nabla_X v|^{p-2}\nabla_X v) \geq H(x)v^{p-1},$$

where $\nabla_X = (X_1, X_2, \dots, X_N)$ is the associated gradient and v is a function satisfying the above hypothesis. From this weighted Hardy type inequality, we recover most of the fundamental Hardy type inequalities including (5.2), (5.3) and (5.4). In Section 5.2, we prove the weighted anisotropic Rellich type inequality for general vector fields.

5.1 Weighted anisotropic Hardy type inequality

In this section, we obtain the weighted anisotropic Hardy type inequalities for general (real-valued) vector fields. It will be proved by using the anisotropic Picone type identity. As consequences, we discover most of the Hardy type inequalities and the uncertainty principles which are known in the setting of the Euclidean space, Heisenberg and Carnot groups.

Consider a family of real vector fields $\{X_k\}_{k=1}^N, N \leq n,$ on a smooth manifold M with dimension n and a volume form dx. Then we say that an open bounded set $\Omega \subset M$ is an admissible domain if its boundary $\partial\Omega$ has no self-intersections, and if the vector fields $\{X_k\}_{k=1}^N$ satisfy

$$\sum_{k=1}^N \int_\Omega X_k f_k dx = \sum_{k=1}^N \int_{\partial\Omega} f_k \langle X_k, dx \rangle, \tag{5.5}$$

for all $f_k \in C^1(\Omega) \cap C(\overline{\Omega}), k = 1, \dots, N$.

First, we formulate an assumption which is important for presenting some examples of Theorem 5.1.3 and of other related results:

Assumption *Let* $T_y \subset M$ *be an open set containing* $y \in M$ *such that the operator*

$$\mathcal{L} := \sum_{i=1}^N X_i^2$$

has a fundamental solution in T_y, *that is, there exists a function* $\Gamma_y \in C^2(T_y \backslash \{y\})$ *such that*

$$-\mathcal{L}\Gamma_y = \delta_y \ in \ T_y, \tag{5.6}$$

where δ_y *is the Dirac* δ*-distribution at* y.

We will say that an admissible domain Ω is a strongly admissible domain with $y \in M$ if the above assumption is satisfied, $\Omega \subset T_y$, and (5.5) holds for $f_k = vX_k\Gamma_y$ for all $v \in C^1(\Omega) \cap C(\overline{\Omega})$.

Note that the fundamental solution for sums of squares of vector fields satisfying Hörmander's condition was obtained by Sánchez-Calle in [75].

Let us recall several important examples from [31] which satisfy the above condition:

Example 1 Let M be a stratified group with $n \geq 3$, and let $\{X_k\}_{k=1}^N$ be the left-invariant vector fields giving the first stratum of M. Then any open bounded set $\Omega \subset M$ with a piecewise smooth simple boundary is strongly admissible.

Example 2 Let $M \equiv \mathbb{R}^n$ with $n \geq 3$, and let the vector fields X_k with $k = 1, \dots, N,\ N \leq n$, have the following form

$$X_k := \frac{\partial}{\partial x_k} + \sum_{m=N+1}^{n} a_{k,m}(x) \frac{\partial}{\partial x_m}, \tag{5.7}$$

where $a_{k,m}(x)$ are locally $C^{1,\alpha}$-regular for some $0 < \alpha \leq 1$, where $C^{1,\alpha}$ stands for the space of functions with X_k-derivative in the Hölder space C^α with respect ot the control distance defined by these vector fields. Assume that

$$\frac{\partial}{\partial x_k} = \sum_{1 \leq i > j \leq N} \lambda^{i,j}(x)[X_i, X_j]$$

for all $k = N+1, \dots, n$ with $\lambda_k^{i,j} \in L_{loc}^\infty(M)$. Then any open bounded set $\Omega \subset M \equiv \mathbb{R}^n$ with a piecewise smooth simple boundary is strongly admissible.

Example 3 More generally, let $M \equiv \mathbb{R}^n$ with $n \geq 3$. Let the vector fields X_k for $k = 1, \dots, N,\ N \leq n$, satisfy the Hörmander commutator condition of step $r \geq 2$. Assume that all the vector fields X_k for $k = 1, \dots, N$ belong to $C^{r,\alpha}(U)$ for some $0 < \alpha \leq 1$ and $U \subset M \equiv \mathbb{R}^n$, and if $r = 2$, then we assume $\alpha = 1$. Then if X_k's are in the form (7), then any open bounded set $\Omega \subset M \equiv \mathbb{R}^n$ with a piecewise smooth simple boundary is strongly admissible.

Moreover, let us recall the following analogue of Green's formulae.

Proposition 5.1.1 [Green's formulae] *Let $\Omega \subset M$ be an admissible domain. Let $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ and $v \in C^1(\Omega) \cap C(\overline{\Omega})$, then we have the following analogue of Green's first formula*

$$\int_{\mathbb{G}} \left((\tilde{\nabla} v)u + v\mathcal{L}u\right)dx = \int_{\partial\Omega} v\langle \tilde{\nabla} u, dx\rangle, \tag{5.8}$$

where

$$\tilde{\nabla} u = \sum_{i=1}^{N} (X_i u)X_i. \tag{5.9}$$

If $u, v \in C^2(\Omega) \cap C^1(\overline{\Omega})$, then we have the following analogue of Green's second *formula*

$$\int_\Omega (u\mathcal{L}v - v\mathcal{L}u)dx = \int_{\partial\Omega} \left(u\langle\tilde{\nabla}v, dx\rangle - v\langle\tilde{\nabla}u, dx\rangle\right). \tag{5.10}$$

First, we present the anisotropic Picone type identity for vector fields.

Lemma 5.1.2 *Let* $\Omega \subset M$ *be an open set. Let* u, v *be differentiable a.e. in* Ω, $v > 0$ *a.e. in* Ω *and* $u \geq 0$. *Define*

$$R(u,v) := \sum_{i=1}^N |X_i u|^{p_i} - \sum_{i=1}^N X_i \left(\frac{u^{p_i}}{v^{p_i-1}}\right) |X_i v|^{p_i-2} X_i v, \tag{5.11}$$

$$L(u,v) := \sum_{i=1}^N |X_i u|^{p_i} - \sum_{i=1}^N p_i \frac{u^{p_i-1}}{v^{p_i-1}} |X_i v|^{p_i-2} X_i v X_i u + \sum_{i=1}^N (p_i - 1) \frac{u^{p_i}}{v^{p_i}} |X_i v|^{p_i}, \tag{5.12}$$

where $p_i > 1$, $i = 1, \ldots, N$. *Then*

$$L(u,v) = R(u,v) \geq 0. \tag{5.13}$$

In addition, we have $L(u,v) = 0$ *a.e. in* Ω *if and only if* $u = cv$ *a.e. in* Ω *with a positive constant* c.

Proof of Lemma 5.1.2. First, we show the equality in (5.13) by a direct computation as follows

$$R(u,v) = \sum_{i=1}^N |X_i u|^{p_i} - \sum_{i=1}^N X_i \left(\frac{u^{p_i}}{v^{p_i-1}}\right) |X_i v|^{p_i-2} X_i v$$
$$= \sum_{i=1}^N |X_i u|^{p_i} - \sum_{i=1}^N p_i \frac{u^{p_i-1}}{v^{p_i-1}} |X_i v|^{p_i-2} X_i v X_i u + \sum_{i=1}^N (p_i - 1) \frac{u^{p_i}}{v^{p_i}} |X_i v|^{p_i}$$
$$= L(u,v).$$

Now we rewrite $L(u,v)$ to see $L(u,v) \geq 0$, that is,

$$L(u,v) = \sum_{i=1}^N |X_i u|^{p_i} - \sum_{i=1}^N p_i \frac{u^{p_i-1}}{v^{p_i-1}} |X_i v|^{p_i-1} |X_i u| + \sum_{i=1}^N (p_i - 1) \frac{u^{p_i}}{v^{p_i}} |X_i v|^{p_i}$$
$$+ \sum_{i=1}^N p_i \frac{u^{p_i-1}}{v^{p_i-1}} |X_i v|^{p_i-2} (|X_i v||X_i u| - X_i v X_i u)$$
$$= S_1 + S_2,$$

where we denote S_1 and S_2 in the following form

$$S_1 := \sum_{i=1}^{N} p_i \left[\frac{1}{p_i} |X_i u|^{p_i} + \frac{p_i - 1}{p_i} \left(\left(\frac{u}{v} |X_i v| \right)^{p_i - 1} \right)^{\frac{p_i}{p_i - 1}} \right]$$
$$- \sum_{i=1}^{N} p_i \frac{u^{p_i - 1}}{v^{p_i - 1}} |X_i v|^{p_i - 1} |X_i u|,$$

and

$$S_2 := \sum_{i=1}^{N} p_i \frac{u^{p_i - 1}}{v^{p_i - 1}} |X_i v|^{p_i - 2} (|X_i v||X_i u| - X_i v X_i u).$$

Since $|X_i v||X_i u| \geq X_i v X_i u$ we have $S_2 \geq 0$. To check that $S_1 \geq 0$ we will use Young's inequality for $a \geq 0$ and $b \geq 0$ stating that

$$ab \leq \frac{a^{p_i}}{p_i} + \frac{b^{q_i}}{q_i}, \tag{5.14}$$

where $p_i > 1, q_i > 1$, and $\frac{1}{p_i} + \frac{1}{q_i} = 1$ for $i = 1, \dots, N$. The equality in (5.14) holds if and only if $a^{p_i} = b^{q_i}$, i.e. if $a = b^{\frac{1}{p_i - 1}}$. By setting $a = |X_i u|$ and $b = \left(\frac{u}{v} |X_i v| \right)^{p_i - 1}$ in (5.14), we get

$$p_i |X_i u| \left(\frac{u}{v} |X_i v| \right)^{p_i - 1} \leq p_i \left[\frac{1}{p_i} |X_i u|^{p_i} + \frac{p_i - 1}{p_i} \left(\left(\frac{u}{v} |X_i v| \right)^{p_i - 1} \right)^{\frac{p_i}{p_i - 1}} \right]. \tag{5.15}$$

This yields $S_1 \geq 0$ which proves that $L(u, v) = S_1 + S_2 \geq 0$. It is easy to check that $u = cv$ implies $R(u, v) = 0$. Now let us show that $L(u, v) = 0$ implies $u = cv$. Due to $u(x) \geq 0$ and $L(u, v)(x_0) = 0, x_0 \in \Omega$, we consider the two cases $u(x_0) > 0$ and $u(x_0) = 0$. For the case $u(x_0) > 0$ we conclude from $L(u, v)(x_0) = 0$ that $S_1 = 0$ and $S_2 = 0$. Then $S_1 = 0$ implies

$$|X_i u| = \frac{u}{v} |X_i v|, \quad i = 1, \dots, N, \tag{5.16}$$

and $S_2 = 0$ implies

$$|X_i v||X_i u| - X_i v X_i u = 0, \quad i = 1, \dots, N. \tag{5.17}$$

The combination of (5.16) and (5.17) gives

$$\frac{X_i u}{X_i v} = \frac{u}{v} = c, \quad \text{with} \quad c \neq 0, \quad i = 1, \dots, N. \tag{5.18}$$

Let us denote $\Omega^* := \{x \in \Omega | u(x) = 0\}$. If $\Omega^* \neq \Omega$, then suppose that $x_0 \in \partial\Omega^*$. Then there exists a sequence $x_k \notin \Omega^*$ such that $x_k \to x_0$. In particular, $u(x_k) \neq 0$, and hence by the first case we have $u(x_k) = cv(x_k)$. Passing to the limit we get $u(x_0) = cv(x_0)$. Since $u(x_0) = 0$ and $v(x_0) \neq 0$, we get that $c = 0$. But then by the first case again, since $u = cv$ and $u \neq 0$ in $\Omega \backslash \Omega^*$, it is impossible that $c = 0$. This contradiction implies that $\Omega^* = \Omega$. The proof of Lemma 5.1.2 is complete.

Now we are ready to obtain the weighted anisotropic Hardy type inequalities for general vector fields by using the anisotropic Picone type identity.

Theorem 5.1.3 *Let $\Omega \subset M$ be an admissible domain. Let $W_i(x) \geq 0$ and $H_i(x) \geq 0$ be functions with $i = 1, \dots, N$, such that for a function $v \in C^1(\Omega) \cap C(\overline{\Omega})$ and $v > 0$ a.e. in Ω, we have*

$$-X_i(W_i(x)|X_i v|^{p_i-2} X_i v) \geq H_i(x) v^{p_i-1}, \quad i = 1, \dots, N. \tag{5.19}$$

Then, for all functions $0 \leq u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ and the positive function $v \in C^1(\Omega) \cap C(\overline{\Omega})$ satisfying (19), we get

$$\begin{aligned} \sum_{i=1}^N \int_\Omega W_i(x)|X_i u|^{p_i} dx \geq \sum_{i=1}^N \int_\Omega H_i(x)|u|^{p_i} dx \\ + \sum_{i=1}^N \int_{\partial\Omega} \frac{u^{p_i}}{v^{p_i-1}} \langle \tilde{\nabla}_i (W_i(x)|X_i v|^{p_i-2} X_i v), dx \rangle, \end{aligned} \tag{5.20}$$

where $\tilde{\nabla}_i f = X_i f X_i$ and $p_i > 1$, for $i = 1, \dots, N$.

Remark 5.1.4 Note that if u vanishes on the boundary $\partial\Omega$ and $p_i = p$, then we have the weighted Hardy type inequalities for general vector fields

$$\int_\Omega W(x)|\nabla_X u|^p dx \geq \int_\Omega H(x)|u|^p dx, \tag{5.21}$$

where $\nabla_X := (X_1, \dots, X_N)$.

Proof of Theorem 5.1.3. Let us give a brief outline of the following proof. We start by using the property of the anisotropic Picone type identity (5.13), then we apply the divergence theorem and the hypothesis (5.19), respectively. At the end, we arrive at (5.20). Thus, we have

$$\begin{aligned} 0 &\leq \int_\Omega \sum_{i=1}^N W_i(x) L(u,v) dx = \int_\Omega \sum_{i=1}^N W_i(x) R(u,v) dx \\ &= \sum_{i=1}^N \int_\Omega W_i(x)|X_i u|^{p_i} dx - \sum_{i=1}^N \int_\Omega X_i \left(\frac{u^{p_i}}{v^{p_i-1}}\right) W_i(x)|X_i v|^{p_i-2} X_i v dx \\ &= \sum_{i=1}^N \int_\Omega W_i(x)|X_i u|^{p_i} dx + \sum_{i=1}^N \int_\Omega \frac{u^{p_i}}{v^{p_i-1}} X_i(W_i(x)|X_i v|^{p_i-2} X_i v) dx \\ &- \sum_{i=1}^N \int_{\partial\Omega} \frac{u^{p_i}}{v^{p_i-1}} \langle \tilde{\nabla}_i (W_i(x)|X_i v|^{p_i-2} X_i v), dx \rangle \\ &\leq \sum_{i=1}^N \int_\Omega W_i(x)|X_i u|^{p_i} dx - \sum_{i=1}^N \int_\Omega H_i(x) u^{p_i} dx \\ &- \sum_{i=1}^N \int_{\partial\Omega} \frac{u^{p_i}}{v^{p_i-1}} \langle \tilde{\nabla}_i (W_i(x)|X_i v|^{p_i-2} X_i v), dx \rangle, \end{aligned}$$

where $\widetilde{\nabla}_i f = X_i f X_i$. This completes the proof of Theorem 5.1.3.

Now we present some concrete examples of the weighted anisotropic Hardy type inequalities (5.20).

Note that examples of the weighted anisotropic Hardy type inequalities on M will be expressed in terms of the fundamental solution $\Gamma = \Gamma_y(x)$ in the **assumption**. For brevity, we can just write it as Γ, if we fix some $y \in M$ and the corresponding T_y and Γ_y.

Corollary 5.1.5 *Let $\Omega \subset M$ be an admissible domain. Let $\alpha \in \mathbb{R}, 1 < p_i < \beta + \alpha, i = 1, \dots, N,$ and $\gamma > -1, \beta > 2$. Then for all $u \in C_0^\infty(\Omega \backslash \{0\})$ we have*

$$\sum_{i=1}^N \int_\Omega \Gamma^{\frac{\alpha}{2-\beta}} |X_i \Gamma^{\frac{1}{2-\beta}}|^\gamma |X_i u|^{p_i} dx \geq \sum_{i=1}^N \left(\frac{\beta+\alpha-p_i}{p_i}\right)^{p_i} \int_\Omega \Gamma^{\frac{\alpha-p_i}{2-\beta}} |X_i \Gamma^{\frac{1}{2-\beta}}|^{p_i+\gamma} |u|^{p_i} dx. \tag{5.22}$$

Note that (5.22) is an analogue of the result of Wang and Niu [73], now for general vector fields.

Remark 5.1.6 By taking $\gamma = 0$ and $p_i = 2$ we have the following inequality

$$\int_\Omega \Gamma^{\frac{\alpha}{2-\beta}} |\nabla_X u|^2 dx \geq \sum_{i=1}^N \left(\frac{\beta+\alpha-2}{2}\right)^2 \int_\Omega \Gamma^{\frac{\alpha-2}{2-\beta}} |\nabla_X \Gamma^{\frac{1}{2-\beta}}|^2 |u|^2 dx, \tag{5.23}$$

for all $u \in C_0^\infty(\Omega)$ and where $\nabla_X = (X_1, \dots, X_N)$.

Proof of Corollary 5.1.5. Consider the functions W_i and v such that

$$W_i = d^\alpha |X_i d|^\gamma \text{ and } v = \Gamma^{\frac{\psi}{2-\beta}} = d^\psi, \tag{5.24}$$

where we denote $d = \Gamma^{\frac{1}{2-\beta}}$ and $\psi = -\left(\frac{\beta+\alpha-p_i}{p_i}\right)$ for simplicity. Now we plug (5.24) in (5.19) to calculate the function H_i. Before we need to have the following computations

$$\begin{aligned}
&X_i v = \psi d^{\psi-1} X_i d, \\
&|X_i v|^{p_i-2} = |\psi|^{p_i-2} d^{(\psi-1)(p_i-2)} |X_i d|^{p_i-2}, \\
&W_i |X_i v|^{p_i-2} X_i v = |\psi|^{p_i-2} \psi d^{\alpha+(\psi-1)(p_i-1)} |X_i d|^{\gamma+p_i-2} X_i d.
\end{aligned}$$

Also, we get

$$\begin{aligned}
&\sum_{i=1}^N X_i^2 d^\alpha = \sum_{i=1}^N X_i (X_i \Gamma^{\frac{\alpha}{2-\beta}}) = \sum_{i=1}^N X_i \left(\frac{\alpha}{2-\beta} \Gamma^{\frac{\alpha+\beta-2}{2-\beta}} X_i \Gamma\right) \\
&= \frac{\alpha(\alpha+\beta-2)}{(2-\beta)^2} \Gamma^{\frac{\alpha+2\beta-4}{2-\beta}} \sum_{i=1}^N |X_i \Gamma|^2 + \frac{\alpha}{2-\beta} \Gamma^{\frac{\alpha+\beta-2}{2-\beta}} \sum_{i=1}^N X_i^2 \Gamma \\
&= \frac{\alpha(\alpha+\beta-2)}{(2-\beta)^2} d^{\alpha+2\beta-4} \sum_{i=1}^N |X_i d^{2-\beta}|^2
\end{aligned}$$

$$= \alpha(\alpha+\beta-2)d^{\alpha-2}\sum_{i=1}^{N}|X_i d|^2. \tag{5.25}$$

We observe that $\sum_{i=1}^{N} X_i^2\Gamma = 0$, since $\Gamma = \Gamma_y$ is the fundamental solution to $\mathcal{L}$. Also, we have

$$\begin{aligned} &X_i|X_i d|^{\gamma} = X_i((X_i d)^2)^{\frac{\gamma}{2}} \\ &= \gamma|X_i d|^{\gamma-2}X_i d X_i^2 d \\ &= \gamma(\beta-1)d^{-1}|X_i d|^{\gamma}X_i d. \end{aligned} \tag{5.26}$$

In the last line, we have used (25) with $\alpha = 1$. Using (5.25) and (5.26) we compute

$$\begin{aligned} &X_i(W_i|X_i v|^{p_i-2}X_i v) = |\psi|^{p_i-2}\psi X_i\big(d^{\alpha+(\psi-1)(p_i-1)}|X_i d|^{\gamma+p_i-2}X_i d\big) \\ &= |\psi|^{p_i-2}\psi\big((\alpha+(\psi-1)(p_i-1))d^{\alpha+(\psi-1)(p_i-1)-1}|X_i d|^{\gamma+p_i}\big) \\ &+|\psi|^{p_i-2}\psi\big((\gamma+p_i-2)(\beta-1)d^{\alpha+(\psi-1)(p_i-1)-1}|X_i d|^{\gamma+p_i}\big) \\ &+|\psi|^{p_i-2}\psi\big((\beta-1)d^{\alpha+(\psi-1)(p_i-1)-1}|X_i d|^{\gamma+p_i}\big) \\ &= |\psi|^{p_i-2}\psi(-\psi+(\gamma+p_i-2)(\beta-1))d^{\alpha-p_i+\psi(p_i-1)}|X_i d|^{\gamma+p_i} \\ &= -|\psi|^{p_i}d^{\alpha-p_i}|X_i d|^{\gamma+p_i}v^{p_i-1} \\ &+|\psi|^{p_i-2}\psi(\gamma+p_i-2)(\beta-1)d^{\alpha-p_i}|X_i d|^{\gamma+p_i}v^{p_i-1}. \end{aligned}$$

Now we put back the value of ψ, then we get

$$\begin{aligned} &-X_i(W_i|X_i v|^{p_i-2}X_i v) = \left|\frac{\beta+\alpha-p_i}{p_i}\right|^{p_i} d^{\alpha-p_i}|X_i d|^{\gamma+p_i}v^{p_i-1} \\ &+\left|\frac{\beta+\alpha-p_i}{p_i}\right|^{p_i-2}\left(\frac{\beta+\alpha-p_i}{p_i}\right)(\gamma+p_i-2)(\beta-1)d^{\alpha-p_i}|X_i d|^{\gamma+p_i}v^{p_i-1} \\ &\geq \left|\frac{\beta+\alpha-p_i}{p_i}\right|^{p_i} d^{\alpha-p_i}|X_i d|^{\gamma+p_i}v^{p_i-1} \\ &\geq H_i(x)v^{p_i-1}. \end{aligned}$$

So we have satisfied the hypothesis, then we plug the values of functions W_i and

$$H_i = \left|\frac{\beta+\alpha-p_i}{p_i}\right|^{p_i}\Gamma^{\frac{\alpha-p_i}{2-\beta}}|X_i\Gamma^{\frac{1}{2-\beta}}|^{\gamma+p_i},$$

in (5.20), which completes the proof.

Corollary 5.1.7 *Let $\Omega \subset M$ be an admissible domain. Let $\alpha,\gamma \in \mathbb{R}$ and $\alpha \neq 0, \beta > 2$. Then for any $u \in C_0^1(\Omega)$ we have*

$$\sum_{i=1}^{N}\int_{\Omega}\Gamma^{\frac{\gamma+p_i}{2-\beta}}|X_i u|^{p_i}dx \geq \sum_{i=1}^{N} C_i(\alpha,\gamma,p_i)^{p_i}\int_{\Omega}\Gamma^{\frac{\gamma}{2-\beta}}|X_i\Gamma^{\frac{1}{2-\beta}}|^{p_i}|u|^{p_i}dx, \tag{5.27}$$

where $C_i(\alpha,\gamma,p_i) := \frac{(\alpha-1)(p_i-1)-\gamma-1}{p_i}$, $p_i > 1$, and $i = 1,\dots,N$.

Note that we recover the result of D'Ambrosio in [74, Theorem 2.7]. Corollary

5.1.7 is proved with the same approach as the previous case by considering the functions

$$W_i = \Gamma^{\frac{\gamma+p_i}{2-\beta}} \text{ and } v = \Gamma^{-\frac{(\alpha-1)(p_i-1)-\gamma-1}{(2-\beta)p_i}}.$$

Corollary 5.1.8 *Let $\Omega \subset M$ be an admissible domain. Let $\alpha \in \mathbb{R}, \beta > 2$, $1 < p_i < \beta + \alpha$ for $i = 1, \dots, N$. Then for all $u \in C_0^\infty(\Omega)$ we have*

$$\sum_{i=1}^N \int_\Omega \Gamma^{\frac{\alpha}{2-\beta}} |X_i u|^{p_i} dx \geq \sum_{i=1}^N C_i(\beta, \alpha, p_i) \int_\Omega \Gamma^{\frac{\alpha}{2-\beta}} \frac{|X_i \Gamma^{\frac{1}{2-\beta}}|^{p_i}}{\left(1+\Gamma^{\frac{p_i}{(p_i-1)(2-\beta)}}\right)^{p_i}} |u|^{p_i} dx, \quad (5.28)$$

where $C_i(\beta, \alpha, p_i) := \left(\frac{\beta+\alpha-p_i}{p_i-1}\right)^{p_i-1} (\beta+\alpha)$.

Note that a Carnot group version of inequality (5.28) was established by Goldstein, Kombe and Yener in [71, P. 2015-2016]. Corollary 5.1.8 is proved with the same approach as the previous cases by considering the functions

$$W_i = \Gamma^{\frac{\alpha}{2-\beta}} \text{ and } v = \left(1 + \Gamma^{\frac{p_i}{(p_i-1)(2-\beta)}}\right)^{-\frac{\beta+\alpha-p_i}{p_i}}.$$

Corollary 5.1.9 *Let $\Omega \subset M$ be an admissible domain. Let $\alpha \in \mathbb{R}, \beta > 2$, $1 < p_i < \beta + \alpha$ for $i = 1, \dots, N$. Then for all $u \in C_0^\infty(\Omega)$ we have*

$$\sum_{i=1}^N \int_\Omega \left(1 + \Gamma^{\frac{p_i}{(p_i-1)(2-\beta)}}\right)^{\alpha(p_i-1)} |X_i u|^{p_i} dx \quad (5.29)$$
$$\geq \sum_{i=1}^N C_i(\beta, p_i, \alpha) \int_\Omega \frac{|X_i \Gamma^{\frac{1}{2-\beta}}|^{p_i}}{\left(1 + \Gamma^{\frac{p_i}{(p_i-1)(2-\beta)}}\right)^{(1-p_i)(1-\alpha)}} |u|^{p_i} dx.$$

where $C_i(\beta, p_i, \alpha) := \beta \left(\frac{p_i(\alpha-1)}{p_i-1}\right)^{p_i-1}$.

Note that Carnot and Euclidean versions of inequality (5.29) were established in [71, P. 2015-2016] and [76], respectively. Corollary 5.1.9 is proved with the same approach as the previous case by considering the functions

$$W_i = \left(1 + \Gamma^{\frac{p_i}{(p_i-1)(2-\beta)}}\right)^{\alpha(p_i-1)} \text{ and } v = \left(1 + \Gamma^{\frac{p_i}{(p_i-1)(2-\beta)}}\right)^{1-\alpha}.$$

Corollary 5.1.10 *Let $\Omega \subset M$ be an admissible domain. Let $\beta > 2$, $a, b > 0$ and $\alpha, \gamma, m \in \mathbb{R}$. If $\alpha\gamma > 0$ and $m \leq \frac{\beta-2}{2}$. Then for all $u \in C_0^\infty(\Omega)$ we have*

$$\int_\Omega \frac{(a+b\Gamma^{\frac{\alpha}{2-\beta}})^\gamma}{\Gamma^{\frac{2m}{2-\beta}}}|\nabla_X u|^2 dx \geq C(\beta,m)^2 \int_\Omega \frac{(a+b\Gamma^{\frac{\alpha}{2-\beta}})^\gamma}{\Gamma^{\frac{2m+2}{2-\beta}}}|\nabla_X \Gamma^{\frac{1}{2-\beta}}|^2|u|^2 dx$$

$$+C(\beta,m)\alpha\gamma b \int_\Omega \frac{(a+b\Gamma^{\frac{\alpha}{2-\beta}})^{\gamma-1}}{\Gamma^{\frac{2m-\alpha+2}{2-\beta}}}|\nabla_X \Gamma^{\frac{1}{2-\beta}}|^2|u|^2 dx, \tag{5.30}$$

where $C(\beta,m):=\frac{\beta-2m-2}{2}$ *and* $\nabla_X=(X_1,\dots,X_N)$.

Note that Carnot and Euclidean version of inequality (5.30) were established in [71, P. 2015-2017] and [77], respectively. Corollary 5.1.10 can be proved with the same approach for $p_i=2,\ i=1,\dots,N,$ as the previous cases by considering the functions

$$W=\frac{(a+b\Gamma^{\frac{\alpha}{2-\beta}})^\gamma}{\Gamma^{\frac{2m}{2-\beta}}} \text{ and } v=\Gamma^{-\frac{\beta-2m-2}{2(2-\beta)}}.$$

Theorem 5.1.3 also implies the following uncertainty principles:

Corollary 5.1.11 *Let* $\Omega\subset M$ *be an admissible domain. Let* $\beta>2$. *Then for all* $u\in C_0^\infty(\Omega)$ *we have*

$$\frac{\beta^2}{4}\left(\int_\Omega |u|^2 dx\right)^2 \leq \left(\int_\Omega |\nabla_X \Gamma^{\frac{1}{2-\beta}}|^{-2}|\nabla_X u|^2 dx\right)\left(\int_\Omega \Gamma^{\frac{2}{2-\beta}}|u|^2 dx\right). \tag{5.31}$$

Proof of Corollary 5.1.11. In Theorem 5.1.3, by letting

$$W(x)=|\nabla_X \Gamma^{\frac{1}{2-\beta}}|^{-2} \text{ and } v=e^{-\alpha\Gamma^{\frac{2}{2-\beta}}},$$

where $\alpha\in\mathbb{R}$, we arrive at

$$-4\alpha^2\int_\Omega \Gamma^{\frac{2}{2-\beta}}|u|^2 dx+2\alpha\beta\int_\Omega |u|^2 dx-\int_\Omega |\nabla_X\Gamma^{\frac{1}{2-\beta}}|^{-2}|\nabla_X u|^2 dx\leq 0.$$

It can be noted that above inequality has the form $a\alpha^2+b\alpha+c\leq 0$ if we denote by

$$a:=-4\int_\Omega \Gamma^{\frac{2}{2-\beta}}|u|^2 dx,$$

$$b:=2\beta\int_\Omega |u|^2 dx,$$

and

$$c := -\int_\Omega |\nabla_X \Gamma^{\frac{1}{2-\beta}}|^{-2} |\nabla_X u|^2 dx.$$

Thus, we have $b^2 - 4ac \le 0$ which proves Corollary 5.1.11.

Corollary 5.1.12 *Let $\Omega \subset M$ be an admissible domain. Let $\beta > 2$. Then for all $u \in C_0^\infty(\Omega)$ we have*

$$\left(\int_\Omega |\nabla_X u|^2 dx\right)\left(\int_\Omega \Gamma^{\frac{2}{2-\beta}} |\nabla_X \Gamma^{\frac{1}{2-\beta}}|^2 |u|^2 dx\right) \ge \frac{\beta^2}{4}\left(\int_\Omega |\nabla_X \Gamma^{\frac{1}{2-\beta}}|^2 |u|^2 dx\right)^2. \quad (5.32)$$

Proof of Corollary 5.1.12. Setting

$$W = 1 \text{ and } v = e^{-\alpha \Gamma^{\frac{2}{2-\beta}}},$$

where $\alpha \in \mathbb{R}$, we have

$$\int_\Omega |\nabla_X u|^2 dx \ge 2\alpha\beta \int_\Omega |\nabla_X \Gamma^{\frac{1}{2-\beta}}|^2 |u|^2 dx - 4\alpha^2 \int_\Omega \Gamma^{\frac{2}{2-\beta}} |\nabla_X \Gamma^{\frac{1}{2-\beta}}|^2 |u|^2 dx.$$

Using the same technique as before we prove Corollary 5.1.12.

Corollary 5.1.13 *Let $\Omega \subset M$ be an admissible domain. Let $\beta > 2$. Then for all $u \in C_0^\infty(\Omega)$ we have*

$$\left(\int_\Omega |\nabla_X u|^2 dx\right)\left(\int_\Omega \Gamma^{\frac{2}{2-\beta}} |\nabla_X \Gamma^{\frac{1}{2-\beta}}|^2 |u|^2 dx\right) \quad (5.33)$$
$$\ge \frac{(\beta-1)^2}{4}\left(\int_\Omega \Gamma^{-\frac{1}{2-\beta}} |\nabla_X \Gamma^{\frac{1}{2-\beta}}|^2 |u|^2 dx\right)^2.$$

We can prove it with the same approach by considering the following pair

$$W = 1 \text{ and } v = e^{-\alpha \Gamma^{\frac{1}{2-\beta}}}.$$

5.2 Weighted anisotropic Rellich type inequality

In this section, we now present the anisotropic (second order) Picone type identity. As a byproduct, we obtain the weighted anisotropic Rellich type inequalities for the general vector fields.

Lemma 5.2.1 *Let $\Omega \subset \mathbb{G}$ be an open set. Let u, v be twice differentiable a.e. in Ω and satisfying the following conditions: $u \ge 0$, $v > 0$, $X_i^2 v < 0$ a.e. in Ω. Let $p_i > 1$, $i = 1, \dots, N$. Then we have*

$$L_1(u, v) = R_1(u, v) \ge 0, \quad (5.34)$$

where

$$R_1(u,v):=\sum_{i=1}^{N}|X_i^2u|^{p_i}-\sum_{i=1}^{N}X_i^2\left(\frac{u^{p_i}}{v^{p_i-1}}\right)|X_i^2v|^{p_i-2}X_i^2v,$$

and

$$\begin{aligned}L_1(u,v):=\sum_{i=1}^{N}|X_i^2u|^{p_i}-\sum_{i=1}^{N}p_i\left(\frac{u}{v}\right)^{p_i-1}X_i^2uX_i^2v|X_i^2v|^{p_i-2}\\+\sum_{i=1}^{N}(p_i-1)\left(\frac{u}{v}\right)^{p_i}|X_i^2v|^{p_i}\\-\sum_{i=1}^{N}p_i(p_i-1)\frac{u^{p_i-2}}{v^{p_i-1}}|X_i^2v|^{p_i-2}X_i^2v\left(X_iu-\frac{u}{v}X_iv\right)^2.\end{aligned}$$

Proof of Lemma 5.1.2. A direct computation gives

$$\begin{aligned}&X_i^2\left(\frac{u^{p_i}}{v^{p_i-1}}\right)=X_i\left(p_i\frac{u^{p_i-1}}{v^{p_i-1}}X_iu-(p_i-1)\frac{u^{p_i}}{v^{p_i}}X_iv\right)\\&=p_i(p_i-1)\frac{u^{p_i-2}}{v^{p_i-2}}\left(\frac{(X_iu)v-u(X_iv)}{v^2}\right)X_iu+p_i\frac{u^{p_i-1}}{v^{p_i-1}}X_i^2u\\&-p_i(p_i-1)\frac{u^{p_i-1}}{v^{p_i-1}}\left(\frac{(X_iu)v-u(X_iv)}{v^2}\right)X_iv-(p_i-1)\frac{u^{p_i}}{v^{p_i}}X_i^2v\\&=p_i(p_i-1)\left(\frac{u^{p_i-2}}{v^{p_i-1}}|X_iu|^2-2\frac{u^{p_i-1}}{v^{p_i}}X_ivX_iu+\frac{u^{p_i}}{v^{p_i+1}}|X_iv|^2\right)\\&+p_i\frac{u^{p_i-1}}{v^{p_i-1}}X_i^2u-(p_i-1)\frac{u^{p_i}}{v^{p_i}}X_i^2v\\&=p_i(p_i-1)\frac{u^{p_i-2}}{v^{p_i-1}}\left(X_iu-\frac{u}{v}X_iv\right)^2+p_i\frac{u^{p_i-1}}{v^{p_i-1}}X_i^2u-(p_i-1)\frac{u^{p_i}}{v^{p_i}}X_i^2v,\end{aligned}$$

which gives the equality in (5.34). By Young's inequality we have

$$\frac{u^{p_i-1}}{v^{p_i-1}}X_i^2uX_i^2v|X_i^2v|^{p_i-2}\le\frac{|X_i^2u|^{p_i}}{p_i}+\frac{1}{q_i}\frac{u^{p_i}}{v^{p_i}}|X_i^2v|^{p_i},\quad i=1,\dots,N,$$

where $p_i>1$ and $q_i>1$ with $\frac{1}{p_i}+\frac{1}{q_i}=1$. Since $X_i^2v<0, i=1,\dots,N,$ we arrive at

$$\begin{aligned}L_1(u,v)\ge\sum_{i=1}^{N}|X_i^2u|^{p_i}+\sum_{i=1}^{N}(p_i-1)\frac{u^{p_i}}{v^{p_i}}|X_i^2v|^{p_i}\\-\sum_{i=1}^{N}p_i\left(\frac{|X_i^2u|^{p_i}}{p_i}+\frac{1}{q_i}\frac{u^{p_i}}{v^{p_i}}|X_i^2v|^{p_i}\right)\\-\sum_{i=1}^{N}p_i(p_i-1)\frac{u^{p_i-2}}{v^{p_i-1}}|X_i^2v|^{p_i-2}X_i^2v\left|X_iu-\frac{u}{v}X_iv\right|^2\end{aligned}$$

$$= \sum_{i=1}^{N} \left(p_i - 1 - \frac{p_i}{q_i}\right) \frac{u^{p_i}}{v^{p_i}} |X_i^2 v|^{p_i}$$
$$- \sum_{i=1}^{N} p_i(p_i - 1) \frac{u^{p_i-2}}{v^{p_i-1}} |X_i^2 v|^{p_i-2} X_i^2 v \left|X_i u - \frac{u}{v} X_i v\right|^2 \geq 0.$$

This completes the proof of Lemma 5.2.1.

Now we are ready to prove the weighted anisotropic Rellich inequalities.

T h e o r e m 5.2.2 *Let $\Omega \subset M$ be an admissible domain. Let $W_i(x) \in C^2(\Omega)$ and $H_i(x) \in L^1_{loc}(\Omega)$ be the nonnegative weight functions. Let $v > 0$, $v \in C^2(\Omega) \cap C^1(\overline{\Omega})$ with*

$$X_i^2\big(W_i(x)|X_i^2 v|^{p_i-2} X_i^2 v\big) \geq H_i(x) v^{p-1}, \; -X_i^2 v > 0, \tag{5.35}$$

a.e. in Ω, for all $i = 1, \dots, N$. Then for every $0 \leq u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ we have the following inequality

$$\sum_{i=1}^{N} \int_{\Omega} H_i(x)|u|^{p_i} dx \leq \sum_{i=1}^{N} \int_{\Omega} W_i(x)|X_i^2 u|^{p_i} dx \tag{5.36}$$
$$+ \sum_{i=1}^{N} \int_{\partial\Omega} W_i(x)|X_i^2 v|^{p_i-2} X_i^2 v \langle \widetilde{\nabla}_i \left(\frac{u^{p_i}}{v^{p_i-1}}\right), dx\rangle$$
$$- \sum_{i=1}^{N} \int_{\partial\Omega} \left(\frac{u^{p_i}}{v^{p_i-1}}\right) \langle \widetilde{\nabla}_i (W_i(x)|X_i^2 v|^{p_i-2} X_i^2 v), dx\rangle,$$

where $1 < p_i < N$ for $i = 1, \dots, N$, and $\widetilde{\nabla}_i u = X_i u X_i$.

Note that a Carnot group version of Theorem 5.2.2 was obtained by Goldstein, Kombe and Yener in [78]. Moreover, it should be also noted that the function v from the assumption (5.35) appears in the boundary terms (5.36), which seems a new effect unlike known particular cases of Theorem 5.2.2.

P r o o f o f T h e o r e m 5.2.2. Let us give a brief outline of the following proof as in Theorem 5.2.2. We start by using the property of the anisotropic (second order) Picone type identity (5.34), then we apply analogue of Green's second formula from Proposition 1 and the hypothesis (5.35), respectively. Finally, we arrive at (5.36) by using $H_i(x) \geq 0$. Thus, we have

$$0 \leq \int_{\Omega} W_i(x) L_1(u, v) dx = \int_{\Omega} W_i(x) R_1(u, v) dx$$
$$= \int_{\Omega} W_i(x)|X_i^2 u|^{p_i} dx - \int_{\Omega} X_i^2 \left(\frac{u^{p_i}}{v^{p_i-1}}\right) W_i(x)|X_i^2 v|^{p_i-2} X_i^2 v dx$$
$$= \int_{\Omega} W_i(x)|X_i^2 u|^{p_i} dx - \int_{\Omega} \frac{u^{p_i}}{v^{p_i-1}} X_i^2\big(W_i(x)|X_i^2 v|^{p_i-2} X_i^2 v\big) dx$$
$$+ \int_{\partial\Omega} \Bigg(W_i(x)|X_i^2 v|^{p_i-2} X_i^2 v \langle \widetilde{\nabla}_i \left(\frac{u^{p_i}}{v^{p_i-1}}\right), dx\rangle$$
$$- \left(\frac{u^{p_i}}{v^{p_i-1}}\right) \langle \widetilde{\nabla}_i (W_i(x)|X_i^2 v|^{p_i-2} X_i^2 v), dx\rangle\Bigg)$$
$$\leq \int_{\Omega} W_i(x)|X_i^2 u|^{p_i} dx - \int_{\Omega} H_i(x)|u|^{p_i} dx$$

$$+\int_{\partial\Omega}\left(W_i(x)|X_i^2 v|^{p_i-2}X_i^2 v\langle\widetilde{\nabla}_i\left(\frac{u^{p_i}}{v^{p_i-1}}\right),dx\rangle\right.$$
$$\left.-\left(\frac{u^{p_i}}{v^{p_i-1}}\right)\langle\widetilde{\nabla}_i(W_i(x)|X_i^2 v|^{p_i-2}X_i^2 v),dx\rangle\right).$$

In the last line, we have used (5.35) which leads to (5.36).

Let us recall that the operator $\mathcal{L}$ is the sum of squares of vector fields, defined by

$$\mathcal{L}:=\sum_{i=1}^{N} X_i^2. \tag{5.37}$$

Corollary 5.2.3 *Let $\Omega\subset M$ be an admissible domain. Let $\beta>2$, $\alpha\in\mathbb{R}$, $\beta+\alpha>4$ and $\beta>\alpha$. Then for all $u\in C_0^\infty(\Omega\backslash\{0\})$ we have*

$$\int_\Omega \frac{\Gamma^{\frac{\alpha}{2-\beta}}}{|\nabla_X\Gamma^{\frac{1}{2-\beta}}|^2}|\mathcal{L}u|^2dx\geq C(\beta,\alpha)\int_\Omega \Gamma^{\frac{\alpha-4}{2-\beta}}|\nabla_X\Gamma^{\frac{1}{2-\beta}}|^2|u|^2dx, \tag{5.38}$$

where $C(\beta,\alpha):=\frac{(\beta+\alpha-4)^2(\beta-\alpha)^2}{16}$ is in general sharp.

Remark 5.2.4 Note that Kombe [68] proved the sharpness of the constant appearing above inequality for the Carnot groups.

Proof of Corollary 5.2.3. Let us choose the function $W(x)$ and v such that

$$W(x)=\frac{\Gamma^{\frac{\alpha}{2-\beta}}}{|X_i\Gamma^{\frac{1}{2-\beta}}|^2}\ \text{ and }\ v=\Gamma^{\frac{\gamma}{2-\beta}}, \tag{5.39}$$

where $\gamma=-\frac{\beta+\alpha-4}{2}$. As in the case of the Hardy inequality, we use the notation $\Gamma=d^{2-\beta}$ for simplicity, then we get

$$\begin{aligned}
\sum_{i=1}^{N} X_i^2 d^\gamma &= \sum_{i=1}^{N} X_i^2\Gamma^{\frac{\gamma}{2-\beta}}=\sum_{i=1}^{N} X_i\left(\frac{\gamma}{2-\beta}\Gamma^{\frac{\gamma+\beta-2}{2-\beta}}X_i\Gamma\right)\\
&=\frac{\gamma(\gamma+\beta-2)}{(2-\beta)^2}\Gamma^{\frac{\gamma+2\beta-4}{2-\beta}}\sum_{i=1}^{N}|X_i\Gamma|^2+\frac{\gamma}{2-\beta}\Gamma^{\frac{\gamma+\beta-2}{2-\beta}}\sum_{i=1}^{N} X_i^2\Gamma\\
&=\frac{\gamma(\gamma+\beta-2)}{(2-\beta)^2}d^{\gamma+2\beta-4}\sum_{i=1}^{N}|X_i d^{2-\beta}|^2\\
&=\gamma(\gamma+\beta-2)d^{\gamma-2}\sum_{i=1}^{N}|X_i d|^2.
\end{aligned}$$

We observe that $\sum_{i=1}^{N} X_i^2\Gamma=0$, since $\Gamma=\Gamma_y$ is the fundamental solution to $\mathcal{L}$. Now we can compute the function $H(x)$,

$$\begin{aligned}
X_i^2\big(W_i(x)X_i^2 v\big)&=X_i^2(\gamma(\gamma+\beta-2)d^{\gamma+\alpha-2})\\
&=\gamma(\gamma+\beta-2)(\gamma+\alpha-2)(\gamma+\alpha+\beta-4)d^{\gamma+\alpha-4}|X_i d|^2.
\end{aligned}$$

By putting back $\gamma = -\frac{\beta+\alpha-4}{2}$ we have

$$\gamma + \beta - 2 = \frac{\beta-\alpha}{2},$$
$$\gamma + \alpha - 2 = -\frac{\beta-\alpha}{2},$$
$$\gamma + \alpha + \beta - 4 = \frac{\beta+\alpha-4}{2}.$$

Then

$$X_i^2\left(W(x)X_i^2 v\right) = \left(\frac{\beta+\alpha-4}{2}\right)^2 \left(\frac{\beta-\alpha}{2}\right)^2 d^{\alpha-4}|X_i d|^2 v$$
$$= H(x)v.$$

So we have the values of functions $W(x)$ and

$$H(x) = \left(\frac{\beta + \alpha - 4}{2}\right)^2 \left(\frac{\beta - \alpha}{2}\right)^2 \Gamma^{\frac{\alpha-4}{2-\beta}}|X_i \Gamma^{\frac{1}{2-\beta}}|^2,$$

which allows to plug them in (5.36) yielding

$$\sum_{i=1}^{N} \left(\frac{\beta + \alpha - 4}{2}\right)^2 \left(\frac{\beta - \alpha}{2}\right)^2 \int_\Omega \Gamma^{\frac{\alpha-4}{2-\beta}}|X_i \Gamma^{\frac{1}{2-\beta}}|^2 |u|^2 dx \leq \sum_{i=1}^{N} \int_\Omega \frac{\Gamma^{\frac{\alpha}{2-\beta}}}{|X_i \Gamma^{\frac{1}{2-\beta}}|^2} |X_i^2 u|^2 dx.$$

Note that the sharpness of the constant was obtained by Kombe [68] in the setting of the Carnot groups. In this general case, the argument is the same.

The following corollary can be also proved with the same approach as the above case by setting

$$W(x) = \frac{\Gamma^{\frac{\alpha+2p-2}{2-\beta}}}{|\nabla_X \Gamma^{\frac{1}{2-\beta}}|^{2p-2}} \text{ and } v = \Gamma^{-\frac{\beta+\alpha-2}{p(2-\beta)}}.$$

Corollary 5.2.5 *Let $\Omega \subset M$ be an admissible domain. Let $1 < p < \infty$ and $2 - \beta < \alpha < min\{(\beta - 2)(p - 1), (\beta - 2)\}$. Then for all $u \in C_0^\infty(\Omega\backslash\{0\})$ we have*

$$\int_\Omega \frac{\Gamma^{\frac{\alpha+2p-2}{2-\beta}}}{|\nabla_X \Gamma^{\frac{1}{2-\beta}}|^{2p-2}} |\mathcal{L}u|^p dx \geq C(\beta, \alpha, p)^p \int_\Omega \Gamma^{\frac{\alpha-2}{2-\beta}}|\nabla_X \Gamma^{\frac{1}{2-\beta}}|^2 |u|^p dx, \quad (5.40)$$

where $C(\beta, \alpha, p) := \frac{(\beta+\alpha-2)}{p}\frac{(\beta-2)(p-1)-\alpha}{p}$ is sharp.

Remark 5.2.6 Note that Lian [79] presented the sharpness of the constant appearing in (5.40) in the case of the Carnot groups.

CONCLUSION

In this PhD thesis, we have presented the new significant results to the homogeneous groups, as well as numerous supporting results we believe are interesting and important in their own right.

Let us review the establishedfb results in this dissertation:

In the first direction, where we study the geometric subelliptic inequalities, we presented L^2 and L^p versions of the (subelliptic) geometric Hardy inequalities in half-spaces and convex domains on general stratified groups. As a consequence, we have derived the Hardy-Sobolev inequality in the half-space on the Heisenberg group. Moreover, the geometric Hardy inequality on the starshaped sets is established.

In the second direction, where we focus on the horizontal subelliptic inequalities, we established the version of horizontal weighted Hardy-Rellich type inequalities on the stratified Lie groups [80], as the result of this inequality Sobolev type spaces are defined on stratified Lie groups and the embedding theorems are proved for these functional spaces [81-82]. Also, we have obtained the subelliptic Picone type identities, as a result, we proved the Hardy and Rellich type inequalities for the anisotropic p-sub-Laplacians [83]. Moreover, analogues of Hardy type inequalities with multiple singularities and many-particle Hardy type inequalities are obtained on the stratified groups.

In the third direction, where we investigate on the subelliptic inequalities with the sub-Laplacian fundamental solution, we obtained the generalised weighted L^p-Hardy, L^p-Rellich, and L^p-Caffarelli-Kohn-Nirenberg type inequalities with boundary terms on the stratified groups. As consequences, most of the Hardy type inequalities and the Heisenberg-Pauli-Weyl type uncertainty principles on the stratified groups are recovered. Moreover, a weighted L^2-Rellich type inequality with the boundary term is obtained. We also present Hardy and Rellich inequalities for the sub-Laplacians in terms of their fundamental solutions on the quaternion Heisenberg group.

In the fourth direction, we established the weighted anisotropic Hardy and Rellich type inequalities with boundary terms for general (real-valued) vector fields. As consequences, we derive new as well as many of the fundamental Hardy and Rellich type inequalities which are known in different settings [84-87].

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